

ISOPERIMETRIC INEQUALITIES FOR THE LEAST HARMONIC MAJORANT OF $|x|^p$

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ABSTRACT. Let D be an open set in the d -dimensional Euclidean space \mathbf{R}^d containing the origin 0 and let $h^{(p)}(x, D)$ be the least harmonic majorant of $|x|^p$ in D , where $0 < p < \infty$ if $d \geq 2$ and $1 \leq p < \infty$ if $d = 1$. We shall be concerned with the following isoperimetric inequalities $h^{(p)}(0, D)^{1/p} \leq cr(D)$, where $r(D)$ denotes the volume radius of D , namely, a ball with radius $r(D)$ has the same volume as D has and c is a constant dependent on d and p but independent of D . We fix d and denote by $c(p)$ the infimum of such constants c . As a function of p , $c(p)$ is nondecreasing and satisfies $c(p) \geq 1$. We shall show

(1) there are positive constants C_1 and C_2 such that $C_1 p^{(d-1)/d} \leq c(p) \leq C_2 p^{(d-1)/d}$ for $p \geq 1$,

(2) $c(p) = 1$ if $p \leq d + 2^{1-d}$.

Many estimations of $h^{(p)}(0, D)$ and their applications are also given.

Introduction. We begin with considering a well known but very interesting inequality. Let f be a holomorphic function defined in the unit disk U in the complex plane satisfying $f(0) = 0$. Then

$$\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{is})|^2 ds \leq \frac{1}{\pi} \int_U |f'(x + iy)|^2 dx dy.$$

The left-hand side is the square of the Hardy norm of f and the right-hand side is the Dirichlet integral of f divided by π . The inequality is easily verified. If f has the power series expansion $f(z) = \sum_{j=1}^{\infty} a_j z^j$, $z = x + iy$, then the left-hand side and the right-hand side are equal to $\sum |a_j|^2$ and $\sum j|a_j|^2$, respectively.

The Dirichlet integral of f is the area of the image of U counting the multiplicity of the map f . In 1972, a very interesting inequality was discovered by Alexander, Taylor and Ullman. They showed that the Dirichlet integral of f can be replaced by area $f(U)$, the area of the image of U . Namely, for the estimation of the Hardy norm, it is not necessary to count the multiplicity. The inequality has many applications, see [A-Ta-U]. Recently Kobayashi [Ko] gave a new proof of the inequality. He applied the following fact: Let D be an open connected set in the w -plane containing 0 and let h be the least harmonic majorant of $|w|^2$ in D . Then $h(0) \leq (1/\pi)(\text{area } D)$. The idea of his proof is very simple. It is "subordination" of functions and was used frequently in studying function theory. Let $D = f(U)$ and

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consider $h \circ f$. Then $|f|^2 \leq h \circ f$ in U and $h \circ f$ is harmonic in U . Hence

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{is})|^2 ds \leq \frac{1}{2\pi} \int_0^{2\pi} (h \circ f)(re^{is}) ds$$

and, by the mean-value property of harmonic functions, the right-hand side is equal to $h(f(0)) = h(0)$. Combining this with the above estimation of $h(0)$, we obtain the Alexander-Taylor-Ullman inequality.

Thus our first motivation of the present study arose from the inequality $h(0) \leq (1/\pi)(\text{area } D)$. This was proved by Pólya and Szegő [Pó-Sz] for simply connected domains and by Payne [Pa] for multiply connected domains. How can we give its simple proof? How can we generalize it?

The second motivation of the problem came from Brownian motion. Let D be an open connected set in \mathbf{R}^d and let X be a Brownian motion in \mathbf{R}^d starting at a point x in D . For each ω , $X_t(\omega)$ is a continuous map from $[0, \infty)$ to \mathbf{R}^d satisfying $X_0(\omega) = x$ and we interpret ω as a path of a particle. Let

$$\tau(\omega) = \inf\{t \geq 0; X_t(\omega) \notin D\}.$$

We call τ the first exit time of the path ω from D . Of course, τ depends on the path and we obtain no information if we watch each path. Some particles exit quickly, others stay in D for a long time. But, if we take into consideration the distribution P_x of paths, a probability measure defined on the space of paths starting at x , then we obtain many interesting results.

We are interested in the mean value of the first exit time, namely,

$$m(x) = E_x \tau = \int \tau(\omega) dP_x(\omega).$$

The problem is to give good estimates of $m(x)$. If D is bounded, then it is known that $\Delta m = -2$ in D and $m(x)$ tends to 0 as x approaches to regular boundary points of D . Hence $dm(x) + |x|^2$ is harmonic in D and it is the least harmonic majorant h of $|x|^2$ in D . By taking the specific point $x = 0$, we have $m(0) = (1/d)h(0)$. Thus the problem is reduced to the estimation of $h(0)$, the same problem as in the first motivation.

In this paper, we are concerned with the estimation of $h^{(p)}(0, D)$, where $h^{(p)}(x, D)$ denotes the least harmonic majorant of $|x|^p$ in an open subset D of \mathbf{R}^d which contains 0. Here $0 < p < \infty$ if $d \geq 2$ and $1 \leq p < \infty$ if $d = 1$, because $\Delta|x|^p = p(p+d-2)|x|^{p-2}$ and we assume that $|x|^p$ is subharmonic in \mathbf{R}^d .

After giving an integral representation of $h^{(p)}(0, D)$, we shall show $h^{(p)}(0, D)^{1/p} \leq r(D)$ for $p \leq d$, where $r(D)$ denotes the volume radius of D , namely, a ball with radius $r(D)$ has the same volume as D has. The isoperimetric inequality is a starting point for our study. Indeed, throughout this paper, we shall discuss the estimation of $c(p)$ satisfying $h^{(p)}(0, D)^{1/p} \leq c(p)r(D)$ for every D . If $p \leq d$, $c(p) = 1$ as mentioned above. For $p \leq 2$, it was proved by Payne [Pa] using essentially the Schwarz symmetrization, but for $p > 2$ the Schwarz symmetrization does not work well and our result is new.

Preliminary arguments written above are given in §1. Unbounded domains are discussed in §2. In §3, we give three estimations of harmonic measures. As a consequence, we show that there is a positive constant C_2 depending only on d such that $c(p) \leq C_2 p^{(d-1)/d}$ for $p \geq 1$. Other estimations of $h^{(p)}(0, D)$ which have

many applications are also given. In §4, we show basic properties of a function $c(p)$ of p . In particular, we show that there is a positive constant C_1 depending only on d such that $c(p) \geq C_1 p^{(d-1)/d}$ for $p \geq 1$.

Extremal domains are investigated in §5. We call a domain D containing 0 and having the same volume as the unit ball has “extremal” if $h^{(p)}(0, D)^{1/p} = c(p)$. There are positive numbers $R(p)$ and $r(p)$ such that $D \subset B_{R(p)}$ and $\text{vol}(B_{r(p)} \setminus D) = 0$ for every extremal domain D , where B_t denotes a ball of radius t centered at the origin. We give estimations of $R(p)$ and $r(p)$. The existence theorem is proved for $d \neq 2$ by using spherical rearrangements due to Baernstein and Taylor [Bae-Ta]. There are many open questions about extremal domains. They are summarized at the end of §5. Are there any $p > d$ satisfying $c(p) = 1$? We show such p do exist in §6. Our conjecture is $c(p) = 1$ for $p \leq d + 2$. We give a supporting fact by using a variational method. In §8, we discuss applications. We give an estimation of the Hardy norm and show the existence of harmonic majorants of $|x|^p$ in some plane domains for every $p > 0$. We improve results due to Alexander, Taylor and Ullman [A-Ta-U], Hansen [Han] and Hansen and Hayman [Han-Hay]. We also give an estimation of the mean exit time of Brownian motion and an estimation of the solutions to the Poisson problem.

NOTATION. Let D be an open set in \mathbf{R}^d , $d \geq 1$. We denote by ∂D the boundary of D in \mathbf{R}^d . We call an open set a domain if it is connected. We denote by B_r a ball centered at the origin with radius r , namely, $B_r = \{x = (x_1, \dots, x_d) \in \mathbf{R}^d; |x| < r\}$, where $|x| = \left(\sum x_j^2\right)^{1/2}$. We denote by $\sigma = \sigma_d$ the surface area of the $(d-1)$ -dimensional unit hypersphere ∂B_1 , by ds the surface area element of ∂B_1 and we set $d\theta = (1/\sigma)ds$. Note that the surface area is the number of boundary points in the case $d = 1$, namely, $\sigma_1 = 2$. We define the volume radius $r(D)$ of an open set D by $r(D) = (\text{vol } D / \text{vol } B_1)^{1/d}$, namely, $B_{r(D)}$ has the same volume as D has.

Let $h^{(p)}(x) = h^{(p)}(x, D)$ be the least harmonic majorant of $|x|^p$ in an open set D . If there are no harmonic majorants of $|x|^p$ in a connected component of D , we set $h^{(p)}(x) = \infty$ in the component. For a bounded open set D , we denote by $H^f(x) = H^f(x, D)$ the solution of the Dirichlet problem with boundary values f . The Dirichlet problem is considered in each connected component of D . We assume that f is a bounded Borel measurable function on ∂D . We note that $h^{(p)} = H^{|\cdot|^p}$ if D is bounded. We denote by χ_E the characteristic function of E , namely, $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \in W \setminus E$. The whole space W will be known from the context. We write $\omega(x, E, D)$ for $H^{\chi_E}(x, D)$, where E is a Borel subset of ∂D . For fixed x and D , it is a measure on ∂D and is called a harmonic measure.

1. Preliminary lemmas and the definition of $c(p)$. In this section we shall show preliminary results which give us a starting point of the present research.

LEMMA 1.1. *Let D be a domain in \mathbf{R}^d whose closure is contained in a ball B_R and let Φ be a function continuous on $[0, R)$ and continuously differentiable in $(0, R)$ such that $\Phi(|x|)$ is subharmonic in B_R . Then the least harmonic majorant h of $\Phi(|x|)$ in D can be expressed as*

$$h(x) = \int_0^R \Phi'(t) u_t(x) dt + \Phi(0),$$

where

$$u_t(x) = \omega(x, (\partial D) \setminus (B_t \cup \partial B_t), D).$$

For the notation of harmonic measures, see the introduction. We note that $\Phi'(t) \geq 0$ in $(0, R)$, because $\Phi(t)$ is the average of the subharmonic function $\Phi(|x|)$ over ∂B_t and it is nondecreasing. We also note that the lemma does not hold for unbounded domains, in general. We shall discuss unbounded domains in §2.

PROOF. Since the right-hand side of the equality, let us denote it by u , is harmonic in D , it is sufficient to show that $u(x)$ tends to $\Phi(|x_0|)$ as $x \in D$ approaches to a regular boundary point x_0 of D . By definition of u_t ,

$$\lim_{x \rightarrow x_0} u_t(x) = \begin{cases} 1, & t < |x_0|, \\ 0, & t > |x_0|, \end{cases}$$

and so, by the Lebesgue convergence theorem,

$$\lim_{x \rightarrow x_0} \int_0^R \Phi'(t) u_t(x) dt = \int_0^{|x_0|} \Phi'(t) dt = \Phi(|x_0|) - \Phi(0).$$

Hence $u(x) \rightarrow \Phi(|x_0|)$ ($x \rightarrow x_0$). Q.E.D.

COROLLARY 1.2. Let D be a bounded domain and let $h^{(p)}(x, D)$ be the least harmonic majorant of $|x|^p$ in D . Then

$$h^{(p)}(x, D) = p \int_0^\infty t^{p-1} u_t(x) dt.$$

LEMMA 1.3. Let D be a domain in \mathbf{R}^d containing the origin. Then

$$h^{(p)}(0, D) \leq \frac{p}{\sigma} \int_D |x|^{p-d} dx,$$

where σ denotes the surface area of the $(d-1)$ -dimensional unit hypersphere ∂B_1 .

PROOF. We may assume that D is bounded. Let $\omega_t(x) = \omega(x, (\partial B_t) \cap D, B_t \cap D)$ and $v_t(x) = \omega(x, (\partial B_t) \cap D, B_t)$. Then $u_t(x) \leq \omega_t(x) \leq v_t(x)$ in $B_t \cap D$ and $v_t(0) = (1/\sigma) \int_{(\partial B_t) \cap D} ds$, where ds denotes the surface area element. Hence, by Corollary 1.2,

$$\begin{aligned} h^{(p)}(0, D) &\leq p \int_0^\infty t^{p-1} v_t(0) dt \\ &= \frac{p}{\sigma} \int_0^\infty t^{p-d} \left\{ \int_{(\partial B_t) \cap D} ds \right\} t^{d-1} dt \\ &= \frac{p}{\sigma} \int_D |x|^{p-d} dx. \quad \text{Q.E.D.} \end{aligned}$$

The equality assertion will be given in Corollary 2.3.

THEOREM 1.4. Let D be a domain in \mathbf{R}^d containing the origin. If $p \leq d$, then

$$h^{(p)}(0, D)^{1/p} \leq r(D),$$

where $r(D)$ denotes the volume radius of D .

The equality assertion will be also given in Corollary 2.3. The inequality in the theorem was first shown by Pólya and Szegő [Pó-Sz, p. 115] for the case $d = p = 2$

and D is simply connected. Multiply connected domains were treated by Payne [Pa] and he proved the inequality for $p = 2$ and arbitrary d , see Bandle [Ban, p. 70]. Kobayashi [Ko] gave a new proof for the case $d = p = 2$.

PROOF. Let $\lambda D = \{\lambda x; x \in D\}$ for $\lambda > 0$. Then $h^{(p)}(x, \lambda D) = \lambda^p h^{(p)}(x/\lambda, D)$ and so $h^{(p)}(0, \lambda D)^{1/p} = \lambda h^{(p)}(0, D)^{1/p}$. Since $r(\lambda D) = \lambda r(D)$, it is sufficient to show that $h^{(p)}(0, D) \leq 1$ for a domain D having the same volume as B_1 has. If $p \leq d$, then $|x|^{p-d} \geq 1$ in B_1 and $|x|^{p-d} \leq 1$ outside B_1 . Therefore

$$\int_D |x|^{p-d} dx \leq \int_{B_1} |x|^{p-d} dx$$

for every domain D having the same volume as B_1 has. Since the right-hand side is equal to σ/p , by Lemma 1.3, $h^{(p)}(0, D) \leq 1$. Q.E.D.

In the theorem, we have assumed that $p \leq d$. How about $p > d$ in the theorem? To make clear these things, we introduce a number $c(p) = c(p, d)$ defined by

$$c(p) = \sup\{h^{(p)}(0, D)^{1/p}; 0 \in D \text{ and } r(D) = 1\}.$$

Then, by definition,

$$h^{(p)}(0, D)^{1/p} \leq c(p)r(D)$$

for every domain D containing 0. It follows that $c(p) \geq 1$ for every p , because $h^{(p)}(x, B_1) = 1$. Theorem 1.4 asserts that $c(p) = 1$ if $p \leq d$.

2. Integral representations of the least harmonic majorants for unbounded domains. Let f be a bounded boundary function of an unbounded domain D , that is, a bounded Borel measurable function defined on the boundary ∂D of D . For the sake of simplicity, we assume that f is nonconstant and continuous and set $m = \inf_{x \in \partial D} f(x)$ and $M = \sup_{x \in \partial D} f(x)$. The argument below is valid for arbitrary bounded functions if we replace definitions of m and M by suitable ones.

Let \bar{H}_R^f (resp. \underline{H}_R^f) be the solution in $B_R \cap D$ to the Dirichlet problem with boundary values f on $\partial(B_R \cap D) \setminus ((\partial B_R) \cap D)$ and M (resp. m) on $(\partial B_R) \cap D$. Then \bar{H}_R^f (resp. \underline{H}_R^f) decreases (resp. increases) and converges to a harmonic function \bar{H}^f (resp. \underline{H}^f) in D as R increases to ∞ . Since $\underline{H}_R^f(x) \leq \underline{H}^f(x) \leq \bar{H}^f(x) \leq \bar{H}_R^f(x)$, $\bar{H}^f(x)$ (resp. $\underline{H}^f(x)$) converges to $f(x_0)$ as $x \in D$ approaches to a regular boundary point x_0 of D . We call a bounded harmonic function u in D a solution to the Dirichlet problem with boundary values f if $u(x) \rightarrow f(x_0)$ ($x \rightarrow x_0$) for every regular boundary point x_0 of D . \bar{H}^f and \underline{H}^f are such solutions.

If $\bar{H}^f = \underline{H}^f$, we denote it by H^f . The following lemma shows that it is a unique solution to the Dirichlet problem:

LEMMA 2.1. *The following conditions on an unbounded domain D are equivalent:*

- (i) *For any given bounded boundary function of D , a solution to the Dirichlet problem is determined uniquely.*
- (ii) *Every bounded harmonic function in D having nonnegative boundary values except a set of capacity zero is nonnegative in D .*
- (iii) *$\bar{H}^f = \underline{H}^f$ for some nonconstant bounded continuous boundary function f of D .*
- (iv) *$\lim_{R \rightarrow \infty} \omega_R(x) = 0$, where $\omega_R(x) = \omega(x, (\partial B_R) \cap D, B_R \cap D)$.*

PROOF. It follows that $\overline{H}_R^f - \underline{H}_R^f = (M - m)\omega_R$ for every R . Hence (iii) and (iv) are equivalent. It is evident that (ii) implies (i) and (i) implies (iv). Assume that (iv) holds and let u be a bounded harmonic function in D having nonnegative boundary values. We shall show that u is nonnegative in D . Let $|u| \leq C$ in D . Then $u + C\omega_R$ has nonnegative boundary values in $B_R \cap D$ and so nonnegative in $B_R \cap D$. Hence $u = \lim_{R \rightarrow \infty} (u + C\omega_R)$ is nonnegative in D . Q.E.D.

For an unbounded domain D satisfying $\lim \omega_R(x) = 0$, let $u_t(x)$ be a unique solution to the Dirichlet problem with boundary values $\chi_{(\partial D) \setminus (B_t \cup \partial B_t)}$. Now we show

PROPOSITION 2.2. *Let Φ be an unbounded function continuous on $[0, \infty)$ and continuously differentiable in $(0, \infty)$ such that $\Phi(|x|)$ is subharmonic in \mathbf{R}^d . Let h be the least harmonic majorant of $\Phi(|x|)$ in D . Then*

$$(1) \quad h(x) \leq \int_0^\infty \Phi'(t)\omega_t(x)dt + \Phi(0),$$

where $\omega_t(x) = \omega(x, (\partial B_t) \cap D, B_t \cap D)$ for $x \in B_t \cap D$ and $= 1$ for $D \setminus B_t$.

(2) *If $\liminf_{R \rightarrow \infty} \Phi(R)\omega_R(x) = 0$ for some x in D , h can be expressed as*

$$h(x) = \int_0^\infty \Phi'(t)u_t(x)dt + \Phi(0).$$

PROOF. First we note that, by the Harnack inequality, $\liminf \Phi(R)\omega_R(x) = 0$ for some x in D implies the same holds for every x in D . Next we note that $\liminf \Phi(R)\omega_R(x) = 0$ implies $\lim \omega_R(x) = 0$. Hence $u_t(x)$ is well defined.

Let h_R be the least harmonic majorant of $\Phi(|x|)$ in $D_R = B_R \cap D$. It is easy to show that h_R increases and converges to h as R increases to ∞ . By Lemma 1.1,

$$h_R(x) = \int_0^R \Phi'(t)\bar{u}_{t,R}(x)dt + \Phi(0)$$

for x in D_R , where $\bar{u}_{t,R}(x) = \omega(x, (\partial D_R) \setminus (B_t \cup \partial B_t), D_R)$. Since $\bar{u}_{t,R}(x) \leq \omega_t(x)$ in D_R ,

$$h_R(x) \leq \int_0^R \Phi'(t)\omega_t(x)dt + \Phi(0)$$

and so, by letting R tend to ∞ , we have (1).

Since, for every fixed $t > 0$, $\bar{u}_{t,R}$ converges to u_t as R tends to ∞ , by the Fatou lemma,

$$h(x) \geq \int_0^\infty \Phi'(t)u_t(x)dt + \Phi(0).$$

On the other hand $\bar{u}_{t,R} \leq u_t + \omega_R$ in D_R and so

$$\begin{aligned} h_R(x) &\leq \int_0^R \Phi'(t)(u_t(x) + \omega_R(x))dt + \Phi(0) \\ &= \int_0^R \Phi'(t)u_t(x)dt + (\Phi(R) - \Phi(0))\omega_R(x) + \Phi(0). \end{aligned}$$

Hence, by letting R tend to ∞ , we have

$$h(x) \leq \int_0^\infty \Phi'(t) u_t(x) dt + \Phi(0)$$

if $\liminf \Phi(R) \omega_R(x) = 0$. Q.E.D.

Now we deal with the problem when the equality in Lemma 1.3 or Theorem 1.4 occurs.

COROLLARY 2.3. *Assume that $h^{(p)}(0, D) < \infty$. Equality holds in Lemma 1.3 or Theorem 1.4 if and only if $D = B_R \setminus E$ for some $R > 0$, where E is a relatively closed subset of B_R with capacity zero.*

REMARK. If $d = 1$, then a set of capacity zero is the empty set. Hence equality holds if and only if D is an interval whose middle point is the origin.

PROOF. If $D = B_R \setminus E$, then equality holds evidently. Assume that equality in Theorem 1.4 holds. Then, by the argument in the proof of Theorem 1.4, we see that equality holds in Lemma 1.3. So we may assume that equality in Lemma 1.3 holds.

Since

$$\frac{p}{\sigma} \int_D |x|^{p-d} dx = p \int_0^\infty t^{p-1} v_t(0) dt$$

and $\omega_t(0) \leq v_t(0)$,

$$\int_0^\infty t^{p-1} \omega_t(0) dt < \infty$$

and so $\liminf_{t \rightarrow \infty} t^p \omega_t(0) = 0$. Hence, by Proposition 2.2,

$$h^{(p)}(0, D) = p \int_0^\infty t^{p-1} u_t(0) dt.$$

From the assumption and $u_t(0) \leq v_t(0)$, we see that $u_t(0) = v_t(0)$ for almost every $t > 0$. If there is a ball B_{t_0} such that $\text{cap } B_{t_0} \setminus D > 0$ and $D \setminus B_{t_0} \neq \emptyset$, then $u_t(0) < \omega_t(0) < v_t(0)$ for every t in a neighborhood of t_0 . Hence D is a domain mentioned in the corollary. Q.E.D.

PROPOSITION 2.4. *Let Φ and h be as in Proposition 2.2. If $h(x) < \infty$ for some x in D , then $\Phi(R) \omega_R(x)$ converges as R tends to ∞ and*

$$\lim \Phi(R) \omega_R(x) \leq h(x) - \Phi(0).$$

PROOF. Let $\underline{h}_R^{\Phi - \Phi(0)}$ be the solution in $D_R = B_R \cap D$ to the Dirichlet problem with boundary values $\Phi - \Phi(0)$ on $\partial(B_R \cap D) \setminus ((\partial B_R) \cap D)$ and 0 on $(\partial B_R) \cap D$. Then

$$h_R = \underline{h}_R^{\Phi - \Phi(0)} + (\Phi(R) - \Phi(0)) \omega_R + \Phi(0) \quad \text{in } D_R,$$

$0 \leq \underline{h}_R^{\Phi - \Phi(0)} \leq h_R - \Phi(0) \leq h - \Phi(0)$ and $\underline{h}_R^{\Phi - \Phi(0)}$ increases as R increases. Hence

$$\Phi(R) \omega_R(x) = h_R(x) - \underline{h}_R^{\Phi - \Phi(0)}(x) + \Phi(0) \omega_R(x) - \Phi(0)$$

converges to a limit not greater than $h(x) - \Phi(0)$. Q.E.D.

COROLLARY 2.5. *Let D be a domain in \mathbf{R}^d , $d \geq 2$. If $h^{(p)}(x, D) < \infty$, then*

$$h^{(p)}(x, D) = p \int_0^\infty t^{p-1} u_t(x) dt.$$

PROOF. Take q so that $0 < q < p$. Then, by Proposition 2.4, $\lim R^p \omega_R(x) < \infty$ and so $\lim R^q \omega_R(x) = 0$. By Proposition 2.2, $h^{(q)}(x, D)$ can be expressed as above replacing p by q . Since $\lim_{q \rightarrow p} h^{(q)}(x, D) = h^{(p)}(x, D)$, for details see Lemma 4.1, we have the desired representation. Q.E.D.

REMARK. If $d = 1$ and $p > 1$, then $h^{(p)}(x, D) = \infty$ for an unbounded interval $D = (-a, \infty)$, where $a > 0$, whereas $p \int_0^\infty t^{p-1} u_t(0) dt = a^p$. If $d = p = 1$, then $h^{(1)}(0, D) = 2a$ whereas $\int_0^\infty u_t(0) dt = a$. Hence Corollary 2.5 does not hold in the case $d = p = 1$.

After the author completed this section, he knew from M. Essén that Essén, Haliste, Lewis and Shea [E-Hal-L-Sh 1 and 2] discussed the existence of harmonic majorants of $\Phi(|x|)$ as well as the equality stated in Proposition 2.2. From their results we know that, in the case $d \geq 3$, if $\lim \omega_t(0) = 0$ and if $p \int t^{p-1} u_t(0) dt < \infty$, then $h^{(p)}(0, D) < \infty$. In the case $d = 2$, the condition on ω_t must be replaced by $\lim(\log t) \omega_t(0) = 0$. In both cases, we obtain the equality stated in Corollary 2.5.

3. Estimations of harmonic measures $\omega_R(x)$ and constants $c(p)$. We recall that the harmonic measure $\omega_R(x)$ for a given domain D is defined by $\omega_R(x) = \omega(x, (\partial B_R) \cap D, B_R \cap D)$. In this section, we shall show three estimations of $\omega_R(x)$: Lemmas 3.1, 3.4 and 3.13. Combining these with the integral representation of $h^{(p)}(0, D)$, we have estimations of $h^{(p)}(0, D)$. Especially, we see that $c(p)$ defined in §1 is finite for every p .

1. The first estimation is very elementary and uses the Harnack inequality and the maximum principle for harmonic functions. First we show

LEMMA 3.1. *Let D be a domain in \mathbf{R}^d having the same volume as B_1 has. Then*

$$\omega_{3R}(x) \leq (CR)^{-d} \omega_R(x)$$

for x in $B_R \cap D$, where $C = (4((3/2)^d - 1)/3)^{1/d}$.

PROOF. Since

$$\sigma \int_{2R}^{3R} v_t(0) t^{d-1} dt = \int_{(B_{3R} \setminus B_{2R}) \cap D} dx < \text{vol } B_1 = \frac{\sigma}{d},$$

there is at least one $t \in [2R, 3R]$ such that $v_t(0) < (1/(3^d - 2^d))R^{-d}$. We apply the Harnack inequality to the positive harmonic function v_t defined in B_t and obtain $v_t(x) \leq 3 \cdot 2^{d-2} v_t(0)$ for x in $B_{t/2}$ and so $v_t(x) \leq (CR)^{-d}$ in $B_{t/2}$. Since $\omega_{3R}(x) \leq \omega_t(x) \leq v_t(x)$ in $B_t \cap D$ and $t/2 \geq R$,

$$\omega_{3R}(x) \leq (CR)^{-d} \omega_{t/2}(x) \leq (CR)^{-d} \omega_R(x)$$

in $B_R \cap D$. Q.E.D.

By using Lemma 3.1 we have

PROPOSITION 3.2. *Let D and C be as in Lemma 3.1. Then*

$$\omega_t(0) \leq \{(C/9)t\}^{-(d/2)\{\log_3(Ct)-1\}}$$

for $t \geq 3/C$.

PROOF. Take n so that $3^n \leq Ct < 3^{n+1}$. Then by using Lemma 3.1 repeatedly, we have

$$\omega_t(0) \leq \omega_{3 \cdot 3^{n-1}/C}(0) \leq 3^{-d(n-1)} \omega_{3^{n-1}/C}(0) \leq 3^{-dn(n-1)/2} \omega_{3/C}(0) \leq 3^{-dn(n-1)/2}.$$

Since $Ct < 3^{n+1}$,

$$n(n-1)/2 > \{\log_3(Ct) - 1\}\{\log_3(Ct) - 2\}/2$$

and so

$$3^{-dn(n-1)/2} \leq \{(C/9)t\}^{-(d/2)\{\log_3(Ct)-1\}}. \quad \text{Q.E.D.}$$

COROLLARY 3.3. *If D has finite volume, then $\lim_{t \rightarrow \infty} t^p \omega_t(0) = 0$ for every p . The constant $c(p)$ is finite for every p .*

PROOF. The first assertion follows from Proposition 3.2 immediately. The second follows from Propositions 2.2 and 3.2. Q.E.D.

2. The second estimation is obtained by solving a differential inequality. In the argument, we assume that $d \geq 2$. The idea is very simple and the estimation has many applications. After the author gave a proof he found that Carleman [C1] used the same idea in the case $d = 2$, not for balls but for half planes.

Assume that $d \geq 2$. For θ with $0 \leq \theta \leq 1$, let $S(\theta)$ denote a spherical cap on ∂B_1 defined by $S(\theta) = \{y \in \partial B_1; a < y_1 \leq 1\}$, where $-1 \leq a \leq 1$ and a is chosen so that $\int_{S(\theta)} d\theta = \theta$. We define a function F on $[0, 1]$ by

$$F(\theta) = 2 \int_{(\partial B_1) \setminus S(\theta)} \frac{d\theta(y)}{|y - e|^d},$$

where $e = (1, 0, \dots, 0)$. Our second estimation is the following:

LEMMA 3.4. *Let D be a domain in \mathbf{R}^d , $d \geq 2$, and let x be a point in D . Then, for every r and R with $|x| < r \leq R$,*

$$\omega_R(x) \leq \left(\exp - \int_r^R \frac{F(\theta_t)}{t} dt \right) \omega_r(x),$$

where $\theta_t = \int_{(\partial B_t) \cap D} d\theta = \theta((\partial B_t) \cap D)$.

PROOF. Let $\Gamma_t = (\partial B_t) \cap D$ and $\delta > 0$. By the maximum principle, we have

$$\omega_t(x) - \omega_{t+\delta}(x) \geq \left(\inf_{\Gamma_t} (\omega_t - \omega_{t+\delta}) \right) \omega_t(x)$$

in $B_t \cap D$. We note that

$$\omega_t - \omega_{t+\delta} = 1 - \omega_{t+\delta} \geq 1 - v_{t+\delta}$$

on Γ_t , where $v_{t+\delta}(\cdot) = \omega(\cdot, (\partial B_{t+\delta}) \cap D, B_{t+\delta})$. We may assume that θ_t is a continuous function of t . By the Poisson formula,

$$1 - v_{t+\delta}(z) = \frac{1}{(t+\delta)\sigma} \int_{(\partial B_{t+\delta}) \setminus \Gamma_{t+\delta}} \frac{(t+\delta)^2 - t^2}{|y - z|^d} ds(y)$$

for z on Γ_t and so

$$1 - v_{t+\delta} \geq (F(\theta_t)/t)\delta + o(\delta)$$

on Γ_t for small $\delta > 0$. Hence

$$(\omega_t(x) - \omega_{t+\delta}(x))/\omega_t(x) \geq (F(\theta_t)/t)\delta + o(\delta)$$

for small $\delta > 0$. Take an increasing sequence $\{t_j\}_{j=0}^k$ of positive numbers so that $t_0 = r$, $t_k = R$ and $t_j - t_{j-1}$ is sufficiently small for each j . Then

$$\log \left(\frac{\omega_r(x)}{\omega_R(x)} \right) = \sum_{j=1}^k \log \left(\frac{\omega_{t_{j-1}}(x)}{\omega_{t_j}(x)} \right) \geq \sum \frac{\omega_{t_{j-1}}(x) - \omega_{t_j}(x)}{\omega_{t_{j-1}}(x)}.$$

Combining this with the above estimation and letting k tend to ∞ , we have

$$\log \left(\frac{\omega_r(x)}{\omega_R(x)} \right) \geq \int_r^R \frac{F(\theta_t)}{t} dt. \quad \text{Q.E.D.}$$

REMARK 1. Let D and x be as in Lemma 3.4. For t with $0 < t < |x|$, let

$$\omega^t(x) = \lim_{R \rightarrow \infty} \omega(x, (\partial B_t) \cap D, (B_R \cap D) \setminus (B_t \cup \partial B_t)).$$

Then, for every r and R with $0 < r \leq R < |x|$,

$$\omega^r(x) \leq \left(\frac{r}{R} \right)^{d-2} \left(\exp - \int_r^R \frac{F(\theta_t)}{t} dt \right) \omega^R(x).$$

A proof is given as above by using the Poisson formula for the exterior of a ball.

REMARK 2. Let D'_t be a connected component of $D_t = B_t \cap D$ containing x and let $\theta'_t = \theta((\partial B_t) \cap \partial D'_t)$. The inequality in Lemma 3.4 holds even if θ_t is replaced with θ'_t .

Next we summarize the properties of the function F .

LEMMA 3.5. *The function F defined before Lemma 3.4 enjoys the following properties:*

- (1) F is continuous and decreasing on $(0, 1]$.
- (2) $F(\theta) \geq 2^{1-d}(1 - \theta)$ on $(0, 1]$.
- (3) $F(\theta) = 2^{1-d}(1 - \theta) + o(1 - \theta)$ as $\theta \rightarrow 1$.
- (4) There is a positive constant C depending only on d such that $F(\theta) \geq C\theta^{-1/(d-1)}$ in a neighborhood of $\theta = 0$.
- (5) If $d = 2$, $F(\theta) = (1/\pi)(1 + \cos \pi\theta)/\sin \pi\theta$ and $F(\theta) = (2/\pi^2)\theta^{-1} + O(\theta)$ as $\theta \rightarrow 0$.

REMARK. If a function F satisfies the inequality in Lemma 3.4, then $F(\theta)$ tends to 0 as θ tends to 1 in the case $d \geq 3$.

PROOF. It is easily verified that F satisfies (1) to (3). To show (4) and (5), we introduce a function Θ defined on $[-1, 1]$ by

$$\Theta(a) = \int_{-1 \leq y_1 \leq a} d\theta(y).$$

Then Θ is a continuous and increasing function satisfying $0 \leq \Theta \leq 1$, $\Theta(a) = 1 - \theta$ and $F(\theta)$ can be expressed as

$$F(\theta) = 2 \int_{-1}^{\Theta^{-1}(1-\theta)} \frac{d\Theta(y_1)}{(2 - 2y_1)^{d/2}}$$

because

$$|y - e|^2 = (y_1 - 1)^2 + \sum_{j=2}^d y_j^2 = (y_1 - 1)^2 + (1 - y_1^2) = 2 - 2y_1.$$

Since $\sigma_d(\Theta(a + \delta) - \Theta(a)) = (\delta/|y - ae|) \times \sigma_{d-1}|y - ae|^{d-2} + o(\delta)$, where $|y| = 1$ and $y_1 = a$, and so $|y - ae|^2 = 1 - a^2$,

$$d\Theta(y_1) = C_1(1 - y_1^2)^{(d-3)/2} dy_1$$

and

$$F(\theta) = 2^{1-d/2} C_1 \int_{-1}^{\Theta^{-1}(1-\theta)} \frac{(1 + y_1)^{(d-3)/2}}{(1 - y_1^2)^{3/2}} dy_1,$$

where $C_1 = \sigma_{d-1}/\sigma_d$.

Setting $y_1 = \sin \varphi$, $-\pi/2 \leq \varphi \leq \pi/2$, we have

$$\theta = 1 - \Theta(a) = C_1 \int_{\varphi_0}^{\pi/2} (\cos \varphi)^{d-2} d\varphi,$$

where φ_0 satisfies $-\pi/2 \leq \varphi_0 \leq \pi/2$ and $\sin \varphi_0 = a$. Since

$$\begin{aligned} \int_{\varphi_0}^{\pi/2} (\cos \varphi)^{d-2} d\varphi &\geq \int_{\varphi_0}^{\pi/2} (\cos \varphi)^{d-2} \sin \varphi d\varphi = \frac{(\cos \varphi_0)^{d-1}}{d-1}, \\ \frac{1}{\cos \varphi_0} &\geq \left(\frac{C_1}{d-1} \right)^{1/(d-1)} \theta^{-1/(d-1)}. \end{aligned}$$

If $\theta < 1/2$, then $a > 0$ and so

$$F(\theta) = F\left(\frac{1}{2}\right) + 2^{1-d/2} C_1 \int_0^a \frac{(1 + y_1)^{d/2}}{(1 - y_1^2)^{3/2}} dy_1.$$

The integral on the right-hand side of the equality is greater than

$$\int_0^{\varphi_0} \frac{1}{(\cos \varphi)^2} d\varphi \geq \int_0^{\varphi_0} \frac{\sin \varphi}{(\cos \varphi)^2} d\varphi = \frac{1}{\cos \varphi_0} - 1.$$

Hence

$$F(\theta) \geq F(1/2) + 2^{1-d/2} C_1 (C_2 \theta^{-1/(d-1)} - 1),$$

where $C_2 = (C_1/(d-1))^{1/(d-1)}$.

If $d \geq 3$, then

$$\int_{-1}^0 \frac{(1 + y_1)^{(d-3)/2}}{(1 - y_1^2)^{3/2}} dy_1 < 1$$

and so $F(1/2) < 2^{1-d/2} C_1$. Therefore, by taking a smaller constant C than $2^{1-d/2} C_1 C_2$, we have $F(\theta) \geq C \theta^{-1/(d-1)}$ in a neighborhood of $\theta = 0$. If $d = 2$, then

$$\begin{aligned} \int_{-1}^{\Theta^{-1}(1-\theta)} \frac{(1 + y_1)^{d/2}}{(1 - y_1^2)^{3/2}} dy_1 &= \int_{-1}^a \frac{(1 + y_1)}{(1 - y_1^2)^{3/2}} dy_1 \\ &= \int_{-\pi/2}^{\varphi_0} \frac{1 + \sin \varphi}{(\cos \varphi)^2} d\varphi = \frac{1 + \sin \varphi_0}{\cos \varphi_0} \end{aligned}$$

and

$$\theta = (1/\pi) \int_{\varphi_0}^{\pi/2} d\varphi = \frac{1}{\pi} \left(\frac{\pi}{2} - \varphi_0 \right).$$

Hence

$$F(\theta) = \frac{1}{\pi} \frac{1 + \sin \varphi_0}{\cos \varphi_0} = \frac{1}{\pi} \frac{1 + \cos \pi \theta}{\sin \pi \theta}$$

and

$$F(\theta) = (2/\pi^2)\theta^{-1} + O(\theta)$$

in a neighborhood of $\theta = 0$. Q.E.D.

PROPOSITION 3.6. *Let D be a domain in \mathbf{R}^d , $d \geq 2$, containing the origin. Then*

$$h^{(p)}(0, D) \leq p \int_0^\infty R^{p-1} \exp \left(- \int_0^R \frac{F(\theta_t)}{t} dt \right) dR,$$

where $\theta_t = \theta((\partial B_t) \cap D)$ and F is a function defined before Lemma 3.4.

PROOF. By Lemma 3.4,

$$\omega_R(0) \leq \exp \left(- \int_0^R \frac{F(\theta_t)}{t} dt \right).$$

Hence the proposition follows from Proposition 2.2. Q.E.D.

We denote by $I^{(p)}(D)$ the right-hand side of the inequality in Proposition 3.6. It is easy to show that $I^{(p)}(\lambda D) = \lambda^p I^{(p)}(D)$ for $\lambda > 0$. Let θ_t^* be the nonincreasing rearrangement of θ_t . It is a function on $[0, \infty)$ which has the same distribution function as θ_t has and is defined by $\theta_t^* = \inf\{a \geq 0; m(a) \leq t\}$, where $m(a)$ is the 1-dimensional Lebesgue measure of $\{t; \theta_t > a\}$. θ_t^* is continuous on the right and so

$$D^* = B_\rho \cup \{x \in \mathbf{R}^d \setminus \{0\}; \Theta^{-1}(1 - \theta_{|x|}^*) < x_1/|x| \leq 1\}$$

is a domain, where Θ denotes a function defined in the proof of Lemma 3.5 and B_ρ is a ball contained in D . Since $\theta_t^* = \theta((\partial B_t) \cap D^*)$ and

$$\int_0^R \frac{F(\theta_t^*)}{t} dt \leq \int_0^R \frac{F(\theta_t)}{t} dt,$$

we see $I^{(p)}(D) \leq I^{(p)}(D^*)$.

By using these notations we have

LEMMA 3.7. *Let D be a domain in \mathbf{R}^d , $d \geq 2$, containing the origin and having finite volume. Then there is a domain \tilde{D} containing the origin and having the same volume as D has such that $I^{(p)}(D) \leq I^{(p)}(\tilde{D})$ and $\tilde{\theta}_t = \theta((\partial B_t) \cap \tilde{D})$ is nonincreasing.*

PROOF. Since

$$\text{vol } D^* = \sigma \int_0^\infty \theta_t^* t^{d-1} dt \leq \sigma \int_0^\infty \theta_t t^{d-1} dt = \text{vol } D,$$

we can choose $\lambda \geq 1$ so that $\text{vol } \lambda D^* = \text{vol } D$. Then $I^{(p)}(\lambda D^*) = \lambda^p I^{(p)}(D^*) \geq I^{(p)}(D^*) \geq I^{(p)}(D)$. Hence $\tilde{D} = \lambda D^*$ is the desired domain. Q.E.D.

In the definition of F we assumed that $d \geq 2$, but the following proposition holds for every $d \geq 1$.

PROPOSITION 3.8. *There is a constant C depending only on d such that $c(p) \leq Cp^{(d-1)/d}$ for $p \geq 1$.*

PROOF. First we deal with the case $d \geq 2$. Let D be a domain containing 0 and having the same volume as B_1 has. We shall apply Proposition 3.6 to estimate $h^{(p)}(0, D)$ and so, by Lemma 3.7, we may assume that θ_t is nonincreasing. Since

$$R^d \theta_R = d \int_0^R \theta_R t^{d-1} dt \leq d \int_0^\infty \theta_t t^{d-1} dt = 1$$

for every nonincreasing θ_t , $\theta_R \leq R^{-d}$ for $R > 0$. By (4) of Lemma 3.5,

$$F(\theta) \geq C_1 \theta^{-1/(d-1)}$$

for θ with $0 < \theta < C_2$ and so

$$F(\theta_R) \geq C_1 R^{d/(d-1)}$$

for $R > C_3 = C_2^{-1/d}$. Hence

$$\int_0^R \frac{F(\theta_t)}{t} dt \geq \int_{C_3}^R \frac{F(\theta_t)}{t} dt = C_4 R^{d/(d-1)} - C_5$$

for $R > C_3$, where C_4 and C_5 are positive constants. By Proposition 3.6,

$$h^{(p)}(0, D) \leq p \int_0^{C_3} R^{p-1} dR + p(\exp C_5) \int_{C_3}^\infty R^{p-1} \exp(-C_4 R^{d/(d-1)}) dR.$$

Since, for positive constants A , B , p and q ,

$$\begin{aligned} \int_A^\infty R^{p-1} \exp(-(BR)^q) dR &= (qB^p)^{-1} \int_{(AB)^q}^\infty t^{p/q-1} \exp(-t) dt \\ &\leq (qB^p)^{-1} \Gamma(p/q), \end{aligned}$$

$$h^{(p)}(0, D) \leq C_3^p + C_6 p C_7^{-p} \Gamma(p(d-1)/d)$$

for some positive constants C_6 and C_7 . Combining the inequality with the Stirling formula on the Γ -function, we have the desired estimate.

Now we discuss the case $d = 1$. We note the following simple fact. If $D \subset B_R$, then $h^{(p)}(x, D) \leq R^p$ and so $h^{(p)}(x, D)^{1/p} \leq R$. In the case $d = 1$, a domain containing 0 and having the same volume as B_1 has is just an open interval containing 0 and having length 2, so D is always contained in B_2 and $h^{(p)}(0, D)^{1/p} \leq 2$. Hence we can take $C = 2$. Q.E.D.

3. The third is given by using the Carleman differential inequality and the Poincaré-Wirtinger type inequality. There are many works on the subject. We refer to Carleman [C2], Tsuji [Ts1] (or [Ts2, Chapter III, §17]) for $d = 2$ case and Huber [Hu] for $d \geq 2$. We also refer to a good survey Haliste [Hal]. If $d = 2$, for a certain simply connected domain and a certain point x in the domain, we have a very nice estimation

$$\omega_R(x) \leq 4 \exp \left(-\frac{1}{2} \int_{|x|}^R \frac{dt}{\theta_t t} \right)$$

by using the method of extremal length, see Hersch [Her], Haliste [Hal] and Fuchs [F].

The main difference between our estimation and that of Tsuji or Huber is how to apply the Poincaré-Wirtinger type inequality. We apply it to ring domains whereas they applied it to spheres. The difference is not so small. We have many domains for which their method does not work well, but ours works well. For applications, see §8.

Before giving the third estimation, we discuss integral inequalities of the Poincaré and Wirtinger type. For the usual attribution “Wirtinger’s inequality”, see Mitrić [M, pp. 141–154].

Let f be a continuously differentiable function on $[0, \pi]$ satisfying $f(0) = f(\pi) = 0$. Then $\int_0^\pi (f')^2 dx \geq \int_0^\pi f^2 dx$. This is called the Poincaré-Wirtinger inequality. Equality holds if and only if f is a constant multiple of $\sin x$. There are many variations of the inequality. For example, if f is a continuously differentiable function on a cube $Q = [0, \pi]^d$ vanishing on ∂Q , then

$$\int_Q |\text{grad } f|^2 dx \geq d \int_Q f^2 dx$$

and equality holds if and only if f is a constant multiple of $\sin x_1 \sin x_2 \cdots \sin x_d$.

More generally, it has a form

$$\int_\Omega |\text{grad } f|^2 dx \geq \alpha \int_\Omega f^2 dx,$$

where Ω is a bounded domain in \mathbf{R}^d and f is a continuously differentiable function in Ω or on $\Omega \cup \partial\Omega$. The function f vanishes on a sufficiently large subset N of Ω or $\Omega \cup \partial\Omega$ and the constant α depends on an appropriate capacity of N and does not depend on f . In what follows we shall show several inequalities of the Poincaré-Wirtinger type and give the constants α explicitly.

The first is a slight modification of the above and is well known, so we omit the proof.

LEMMA 3.9. *Let f be a continuously differentiable function on $[0, \delta]$ vanishing at least at one point on $[0, \delta]$. Then*

$$\int_0^\delta (f')^2 dx \geq \left(\frac{\pi}{2\delta}\right)^2 \int_0^\delta f^2 dx.$$

For $y = (y_1, \dots, y_d) \neq 0$, let P_y be the orthogonal projection to a hyperplane $y_1 x_1 + \cdots + y_d x_d = 0$.

PROPOSITION 3.10. *Let Ω be a convex domain with diameter δ . Let f be a continuously differentiable function in Ω and let $N = \{x \in \Omega; f(x) = 0\}$. If there are a finite number of nonzero vectors $y^{(1)}, \dots, y^{(n)}$ satisfying*

$$\Omega \subset \bigcup_{j=1}^n P_{y^{(j)}}^{-1}(P_{y^{(j)}}(N)),$$

then

$$\int_\Omega |\text{grad } f|^2 dx \geq \frac{1}{n} \left(\frac{\pi}{2\delta}\right)^2 \int_\Omega f^2 dx.$$

PROOF. Let $e = (1, 0, \dots, 0)$. Then $P_e(x) = (0, x_2, \dots, x_d)$. If $(0, x_2, \dots, x_d) \in P_e(N)$, by Lemma 3.9,

$$\int_L \left(\frac{\partial f}{\partial x_1} \right)^2 dx_1 \geq \left(\frac{\pi}{2\delta} \right)^2 \int_L f^2 dx_1,$$

where $L = \Omega \cap P_e^{-1}((0, x_2, \dots, x_d))$. Hence,

$$\int_{\Omega} |\text{grad } f|^2 dx \geq \int_{N_e} \left(\frac{\partial f}{\partial x_1} \right)^2 dx \geq \left(\frac{\pi}{2\delta} \right)^2 \int_{N_e} f^2 dx,$$

where $N_e = \Omega \cap P_e^{-1}(P_e(N))$. The inequality with e replaced by y holds for every nonzero y and so

$$n \int_{\Omega} |\text{grad } f|^2 dx \geq \left(\frac{\pi}{2\delta} \right)^2 \sum_{j=1}^n \int_{N_{y(j)}} f^2 dx \geq \left(\frac{\pi}{2\delta} \right)^2 \int_{\Omega} f^2 dx. \quad \text{Q.E.D.}$$

Next we treat other projections in the case $d = 2$.

PROPOSITION 3.11. *Let f be a continuously differentiable function in an annulus $A = \{x \in \mathbf{R}^2; r < |x| < R\}$ and let $N = \{x \in A; f(x) = 0\}$. It follows that*

(1) *If $\{x/|x|; x \in N\} = \partial B_1$, then*

$$\int_A |\text{grad } f|^2 dx \geq \frac{r}{R} \left(\frac{\pi}{2(R-r)} \right)^2 \int_A f^2 dx.$$

(2) *If $\{|x|; x \in N\} = (r, R)$, then*

$$\int_A |\text{grad } f|^2 dx \geq \frac{1}{4R^2} \int_A f^2 dx.$$

PROOF. For each fixed s , by Lemma 3.9,

$$\int_r^R \left(\frac{\partial f}{\partial t} \right)^2 (te^{is}) dt \geq \left(\frac{\pi}{2(R-r)} \right)^2 \int_r^R f^2(te^{is}) dt.$$

Since

$$|\text{grad } f|^2 = \left(\frac{\partial f}{\partial t} \right)^2 + \left(\frac{1}{t} \frac{\partial f}{\partial s} \right)^2 \quad \text{in } A,$$

$$\int_r^R \left(\frac{\partial f}{\partial t} \right)^2 t dt \geq r \int_r^R \left(\frac{\partial f}{\partial t} \right)^2 dt$$

and

$$\int_r^R f^2 dt \geq \frac{1}{R} \int_r^R f^2 t dt,$$

we have the desired inequality in (1).

To show (2), we fix t with $r < t < R$ and apply the Poincaré-Wirtinger inequality. Then

$$\int_0^{2\pi} \left(\frac{\partial f}{\partial s} \right)^2 (te^{is}) ds \geq \frac{1}{4} \int_0^{2\pi} f^2(te^{is}) ds.$$

Multiplying both sides by t and integrating both sides from r to R , we obtain

$$R^2 \int_A \left(\frac{1}{t} \frac{\partial f}{\partial s} \right)^2 dx \geq \int_A \left(\frac{\partial f}{\partial s} \right)^2 dx \geq \frac{1}{4} \int_A f^2 dx.$$

Hence (2) follows. Q.E.D.

REMARK. If we replace A by $A(\theta) = \{te^{is} \in \mathbf{R}^2; r < t < R, -\pi\theta < s < \pi\theta\}$, where $0 < \theta \leq 1$, the same inequality holds by replacing A by $A(\theta)$ in (1). In (2) we replace $1/(4R^2)$ by $1/(16\theta^2 R^2)$. Note that $A \neq A(1)$.

We return to an arbitrary $d \geq 1$ and show

PROPOSITION 3.12. *Let Ω , δ , f and N be as in Proposition 3.10. Then*

$$\int_{\Omega} |\text{grad } f|^2 dx \geq \frac{\pi^2}{2} \cdot \frac{\text{vol } N}{\text{vol } B_{\delta}} \cdot \frac{1}{\delta^2} \int_{\Omega} f^2 dx.$$

PROOF. We use notations in the proof of Proposition 3.10. We have proved

$$\int_{\Omega} |\text{grad } f|^2 dx \geq \left(\frac{\pi}{2\delta}\right)^2 \int_{N_y} f^2 dx$$

for every nonzero y . Hence

$$\begin{aligned} \frac{\sigma}{2} \int_{\Omega} |\text{grad } f|^2 dx &\geq \left(\frac{\pi}{2\delta}\right)^2 \int_{S[1/2]} \left\{ \int_{N_y} f^2 dx \right\} ds(y) \\ &= \left(\frac{\pi}{2\delta}\right)^2 \int_{\Omega} \left\{ \int_{S[1/2]} \chi_{N_y}(x) ds(y) \right\} f^2 dx, \end{aligned}$$

where $S[1/2] = \{y \in \partial B_1; y_1 \geq 0\}$ and χ_{N_y} denotes the characteristic function of N_y .

Set $L_x(y) = \{x + ty \in \Omega; t \in \mathbf{R}\}$ and $S_x = \{y \in S[1/2]; L_x(y) \cap N \neq \emptyset\}$. Since $N \subset \bigcup_{y \in S_x} L_x(y)$ for every x in Ω ,

$$\text{vol } N \leq \frac{\delta^d}{d} \int_{S_x} ds(y) = \frac{\delta^d}{d} \int_{S[1/2]} \chi_{N_y}(x) ds(y)$$

for every x in Ω . Combining these inequalities, we have the desired one. Q.E.D.

REMARK. Propositions 3.10–3.12 are valid if Ω or A is replaced by its closure.

Now we show the third estimation of harmonic measure $\omega_R(0)$. We extend $\omega_R(x)$ onto B_R by putting 0 on $B_R \setminus D$. Then ω_R , we consider $\limsup_{y \rightarrow x} \omega_R(y)$ if necessary, is nonnegative subharmonic in B_R and vanishes almost everywhere on $B_R \setminus D$. In what follows we use only these properties of ω_R . For example, the argument is valid for ω_t , $t \geq R$, and u_t , $t \geq R$, if it is well defined, see before Proposition 2.2. So we write u for ω_R and assume that u is nonnegative subharmonic in B_R .

Set

$$M(t) = \frac{1}{t^{d-1}\sigma} \int_{\partial B_t} u^2 ds = \int_{\partial B_t} u^2 d\theta$$

for t in $(0, R)$. Since u^2 is also subharmonic in B_R , M satisfies (1) $0 \leq M < \infty$, (2) $\lim_{t \rightarrow 0} M(t) = u^2(0)$, (3) M is nondecreasing and is a convex function of r^{2-d} , $-\log r$ if $d = 2$. By using these notations we show

LEMMA 3.13. *Let u and M be as above. If u satisfies*

$$\int_{A_t} |\text{grad } u|^2 dx \geq \alpha(t) \int_{A_t} u^2 dx$$

for every t with $r < t < 2r < R/2$, where $A_t = \{x \in \mathbf{R}^d; t < |x| < 2t\}$, then

$$M(r) \leq \left(C \int_r^{2r} \alpha(t) t dt + 1 \right)^{-1} M(4r),$$

where $C = (2^d - 1)/(2^{d-3}d)$.

PROOF. By an approximation process, we may assume that u is sufficiently smooth. In this case M is differentiable and

$$M'(t) = 2 \int_{\partial B_t} u \frac{\partial u}{\partial t} d\theta = \frac{2}{t^{d-1}\sigma} \int_{\partial B_t} u \frac{\partial u}{\partial n} ds,$$

where $\partial u / \partial n$ denotes the exterior normal derivative of u on ∂B_t . Hence, by the Green formula,

$$M'(t) = \frac{2}{t^{d-1}\sigma} \int_{B_t} (u \Delta u + |\text{grad } u|^2) dx.$$

Since $u \geq 0$ and $\Delta u \geq 0$ in B_t , the right-hand side is not less than

$$\frac{2}{t^{d-1}\sigma} \int_{B_t} |\text{grad } u|^2 dx.$$

Hence, by taking $2t$ for t , we have

$$M'(2t) \geq \frac{2}{(2t)^{d-1}\sigma} \int_{B_{2t}} |\text{grad } u|^2 dx \geq \frac{2}{(2t)^{d-1}\sigma} \int_{A_t} |\text{grad } u|^2 dx.$$

By the assumption, we have

$$M'(2t) \geq \frac{2\alpha(t)}{(2t)^{d-1}\sigma} \int_{A_t} u^2 dx.$$

Since $M(t)$ is nondecreasing,

$$\int_{A_t} u^2 dx = \sigma \int_t^{2t} M(t) t^{d-1} dt \geq \sigma M(r) \int_t^{2t} t^{d-1} dt$$

and so

$$M'(2t) \geq (C/2)M(r)\alpha(t)t$$

for every t with $r < t < 2r$. Hence

$$M(4r) - M(2r) = 2 \int_r^{2r} M'(2t) dt \geq CM(r) \int_r^{2r} \alpha(t) t dt$$

and

$$M(4r) \geq CM(r) \int_r^{2r} \alpha(t) t dt + M(r). \quad \text{Q.E.D.}$$

REMARK. From the proof, we know that, to show Lemma 3.13, it is sufficient to assume that

$$\int_{A_t} |\text{grad } u|^2 dx \geq \alpha(t) \int_{B_{bt} \setminus B_{at}} u^2 dx,$$

where $1 \leq a < b \leq 2$. In this case we take $C = (b^d - a^d)/(2^{d-3}d)$. We shall use this argument in Example 3, §8.

Finally we show the following estimation of $h^{(p)}(0, D)$:

PROPOSITION 3.14. *Let D be a domain containing 0. If, for each $t \geq 1$,*

$$\int_{A_t} |\operatorname{grad} f|^2 dx \geq \alpha(t) \int_{A_t} f^2 dx$$

for every continuously differentiable function f in A_t which vanishes almost everywhere on $A_t \setminus D$, where $A_t = \{x \in \mathbf{R}^d; t < |x| < 2t\}$, then

$$h^{(p)}(0, D) \leq 4^p \left\{ 1 + (4^p - 1) \sum_{j=0}^{\infty} 4^{pj} \prod_{k=0}^j \left(C \int_{4^k}^{2 \cdot 4^k} \alpha(t) t dt + 1 \right)^{-1/2} \right\},$$

where C denotes a constant given in Lemma 3.13.

PROOF. To estimate $\omega_R(0)$ for $R > 4$, take j so that $4^{j+1} < R \leq 4^{j+2}$. Since ω_R vanishes almost everywhere on $A_t \setminus D$ and $\omega_R \leq 1$ in B_R , by applying Lemma 3.13 for $u = \omega_R$ repeatedly, we have

$$M(1) \leq \prod_{k=0}^j \left(C \int_{4^k}^{2 \cdot 4^k} \alpha(t) dt + 1 \right)^{-1}$$

and so

$$\omega_R(0) \leq \prod_{k=0}^j \left(C \int_{4^k}^{2 \cdot 4^k} \alpha(t) t dt + 1 \right)^{-1/2}.$$

By Proposition 2.2,

$$h^{(p)}(0, D) \leq p \int_0^4 R^{p-1} dR + p \sum_{j=0}^{\infty} \int_{4^{j+1}}^{4^{j+2}} R^{p-1} \omega_R(0) dR$$

and we have the desired estimation. Q.E.D.

COROLLARY 3.15. *If there is a number j_0 such that, for every $j > j_0$, $\alpha(t) \geq (2/(3C))16^{p-j}$ on $[4^j, 2 \cdot 4^j]$, then $h^{(p)}(0, D) < \infty$.*

PROOF. From the assumption, we have

$$C \int_{4^k}^{2 \cdot 4^k} \alpha t dt + 1 \geq 16^p + 1$$

for $k > j_0$. Take $q > p$ so that $16^q = 16^p + 1$. Then

$$4^{pj} \prod_{k=0}^j \left(C \int_{4^k}^{2 \cdot 4^k} \alpha t dt + 1 \right)^{-1/2} \leq \left\{ 4^{qj_0} \prod_{k=0}^{j_0} \right\} 4^{(p-q)j}$$

for $j > j_0$. Since $4^{p-q} < 1$, by Proposition 3.14, we see that $h^{(p)}(0, D) < \infty$. Q.E.D.

4. $c(p)$ as a function of p . In §3, we have proved that $c(p)$ is finite and satisfies $c(p) \leq Cp^{(d-1)/d}$ for $p \geq 1$. In this section we shall show more about $c(p)$ as a function of p . First we show the following lemma for a fixed domain:

LEMMA 4.1. *Let D be a domain and let x be a point in D , and fix them. Set $\gamma(p) = h^{(p)}(x, D)^{1/p}$. Then*

- (1) $\gamma(p)$ is a nondecreasing function of p .
- (2) $\log \gamma(1/t)$ is a convex function of t , t varies on $(0, 1]$ if $d = 1$ and on $(0, \infty)$ if $d \geq 2$.

PROOF. (1) Since $\Delta h^a = a(a-1)h^{a-2} \sum (\partial h / \partial x_j)^2 + ah^{a-1} \Delta h$ for a constant $a > 0$ and a function $h > 0$, $\Delta h^a \geq 0$ if $a \geq 1$ and $\Delta h \geq 0$. Hence $(h^{(p)})^{q/p}$ is subharmonic in D if $q \geq p$. If D is bounded, then both $(h^{(p)})^{q/p}$ and $h^{(q)}$ have the same boundary values $|x|^q$ except a set of capacity zero and so $(h^{(p)})^{q/p} \leq h^{(q)}$ in D . Hence, by taking an exhaustion of D if necessary, we have $(h^{(p)})^{1/p} \leq (h^{(q)})^{1/q}$ for $q \geq p$.

(2) It is sufficient to show that

$$(1/p + 1/q) \log h^{(2/(1/p+1/q))} \leq (1/p) \log h^{(p)} + (1/q) \log h^{(q)}$$

in D and the inequality follows if we show

$$s = \exp \left(\frac{p+q}{p} \log h^{(2/(1/p+1/q))} - \frac{q}{p} \log h^{(p)} \right)$$

is subharmonic in D . Since $\Delta \exp f = \left(\sum f_{x_j}^2 + \Delta f \right) \exp f$, where $f_{x_j} = \partial f / \partial x_j$, and $\Delta \log h = (h \Delta h - \sum h_{x_j}^2) / h^2$, setting $X_j = h_{x_j}^{(2/(1/p+1/q))} / h^{(2/(1/p+1/q))}$ and $Y_j = h_{x_j}^{(p)} / h^{(p)}$, we have

$$\begin{aligned} \frac{\Delta s}{s} &= \sum \left(\frac{p+q}{p} X_j - \frac{q}{p} Y_j \right)^2 - \sum \left(\frac{p+q}{p} X_j^2 - \frac{q}{p} Y_j^2 \right) \\ &= \frac{q(p+q)}{p^2} \sum (X_j - Y_j)^2 \end{aligned}$$

and so s is subharmonic in D . Q.E.D.

Now we show the properties of $c(p)$.

THEOREM 4.2. *It follows that*

- (1) $c(p) \geq 1$ for every p and $c(p) = 1$ if $p \leq d$.
- (2) There exist positive constants C_1 and C_2 depending only on d such that

$$C_1 p^{(d-1)/d} \leq c(p) \leq C_2 p^{(d-1)/d}$$

for $p \geq 1$. If $d = 1$, we can take $C_1 = 1$ and $C_2 = 2$.

- (3) $c(p)$ is a nondecreasing function of p .

(4) $\log c(1/t)$ is a nonnegative convex function of t , t varies on $(0, 1]$ if $d = 1$ and on $(0, \infty)$ if $d \geq 2$.

PROOF. We have mentioned (1) at the end of §1. Properties (3) and (4) follow immediately from Lemma 4.1. In Proposition 3.8, we have proved $c(p) \leq C_2 p^{(d-1)/d}$ for $p \geq 1$ and that we can take $C_2 = 2$ in the case $d = 1$. What we have to do is to show $c(p) \geq C_1 p^{(d-1)/d}$ for $p \geq 1$ in the case $d \geq 2$. We shall show it by constructing a domain D satisfying $h^{(p)}(0, D)^{1/p} \geq C_1 p^{(d-1)/d}$.

Let $B'_r = \{x' = (x_2, \dots, x_d); |x'| < r\}$ and let $D = (-r, R-r) \times B'_r$, where $R > r$. For given large R we take r so that $\text{vol } D = \text{vol } B_1$, namely, $r = C_3 R^{-1/(d-1)}$,

where $C_3 = (d-1)\sigma_d/(d\sigma_{d-1})$. Let $\omega(x) = \omega(x, \{R-r\} \times B'_r, D)$. We shall take a subharmonic function s in D having a form $s(x) = f(x_1)g(x')$ such that $\omega \geq s$ in D . Since $h^{(p)}(0, D) \geq (R-r)^p \omega(0) \geq (R-r)^p s(0)$, we get $h^{(p)}(0, D)^{1/p} \geq (R-r)s(0)^{1/p}$.

Let $g(x') = 1 - C_4|x'|^2 r^{-2}$ on $\{|x'| \leq r/2\}$ and be equal to a solution to the Dirichlet problem in $\{r/2 < |x'| < r\}$ with boundary values $g(x')$ on $\{|x'| = r/2\}$ and 0 on $\{|x'| = r\}$. We choose a positive constant C_4 depending only on d so that g is continuously differentiable in B'_r .

Let $f(x_1) = (\exp(ax_1) - \exp(-ar))/(\exp a(R-r) - \exp(-ar))$ for $a > 0$. Then $f(-r) = 0$, $f(R-r) = 1$ and $f'' = a^2 f + b$ for a suitable constant $b > 0$. Since

$$\Delta s = \Delta(fg) = f''g + f\Delta'g = f(a^2g + \Delta'g) + bg$$

in D in the sense of distribution, where Δ' denotes the Laplacian in B'_r , and since $\Delta'g = -2(d-1)C_4r^{-2}$ in $\{|x'| < r/2\}$, s is subharmonic in D if a is taken so large that $a^2g - 2(d-1)C_4r^{-2} \geq 0$ in $\{|x'| < r/2\}$. Since $g = 1 - C_4/4$ on $\{|x'| = r/2\}$, we take $a = C_5r^{-1}$, where $C_5 = (2(d-1)C_4/(1 - C_4/4))^{1/2}$. Thus we have a subharmonic function $s(x) = f(x_1)g(x')$ in D satisfying $\omega \geq s$ in D .

Since $s(0) = f(0) = (\exp(ar) - 1)/(\exp(aR) - 1) \geq ar/\exp(aR)$, $ar = C_5$ and $aR = (C_5/C_3)R^{d/(d-1)}$,

$$(R-r)s(0)^{1/p} \geq C_5^{1/p}(R-r)\exp(-(C_5/C_3)R^{d/(d-1)}/p).$$

Take R so that $p = R^{d/(d-1)}$ and assume that p is sufficiently large. Then $R = p^{(d-1)/d}$ and

$$(R-r)s(0)^{1/p} \geq C_1p^{(d-1)/d}$$

for a positive constant C_1 . Q.E.D.

COROLLARY 4.3. *If $c(p) = c(q)$ for $p < q$, then $c(p) = c(q) = 1$.*

PROOF. Assume that $c(q) > 1$. Then, by (1) of Theorem 4.2, $1 < p < q$. Let $s = 1/p$ and $t = 1/q$. Then $t < s < 1$. By (4) of Theorem 4.2,

$$\log c\left(\frac{1}{s}\right) \leq \frac{1-s}{1-t} \log c\left(\frac{1}{t}\right) + \frac{s-t}{1-t} \log c(1) < \log c\left(\frac{1}{t}\right),$$

because $(1-s)/(1-t) < 1$, $c(1/t) = c(q) > 1$ and $c(1) = 1$. This is a contradiction. Q.E.D.

5. Extremal domains. We call a domain D containing the origin and satisfying $\text{vol } D = \text{vol } B_1$ "extremal" if $h^{(p)}(0, D)^{1/p} = c(p)$, for the definition of $c(p)$, see §1. We have seen in Corollary 2.3 that there exists a unique extremal domain except a set of capacity zero and it is the unit ball if $p \leq d$. How about the case $p > d$?

In this section we deal with extremal domains. First we note that if $c(p) > 1$, the extremal domain cannot be the unit ball. A related fact will be shown in Proposition 5.1.

Next we introduce numbers $R(p)$ and $r(p)$. Let R be a positive number satisfying the following condition: For every domain D satisfying $0 \in D$, $\text{vol } D = \text{vol } B_1$ and $D \setminus B_R \neq \emptyset$, there exists a domain \tilde{D} such that $0 \in \tilde{D}$, $\text{vol } \tilde{D} = \text{vol } B_1$, $\tilde{D} \subset B_R$ and $h^{(p)}(0, \tilde{D}) > h^{(p)}(0, D)$. We denote by $R(p)$ the infimum of such R . By definition,

$D \subset B_{R(p)}$ for every extremal domain D . If there are extremal domains, then $R(p)$ is the infimum of R satisfying $D \subset B_R$ for every extremal domain D .

For the definition of $r(p)$, let r be a positive number satisfying the following condition: For every domain D satisfying $0 \in D$, $\text{vol } D = \text{vol } B_1$ and $\text{vol}(B_r \setminus D) > 0$, there exists a domain \tilde{D} such that $0 \in \tilde{D}$, $\text{vol } \tilde{D} = \text{vol } B_1$, $B_r \subset \tilde{D}$ and $h^{(p)}(0, \tilde{D}) > h^{(p)}(0, D)$. We denote by $r(p)$ the supremum of such r . By definition, $\text{vol}(B_{r(p)} \setminus D) = 0$ for every extremal domain D . If there are extremal domains, then $r(p)$ is the supremum of R satisfying $\text{vol}(B_r \setminus D) = 0$ for every extremal domain D .

It is easy to show that $r(p) \leq 1 \leq c(p) \leq R(p)$ for every p . In Proposition 5.6, we shall give an estimation of $R(p)$ and, in Proposition 5.9, that of $r(p)$. In the definition of $r(p)$, we can replace $\text{vol}(B_r \setminus D)$ by $\text{cap}(B_r \setminus D)$. Indeed, if $\text{vol}(B_r \setminus D) = 0$ but $\text{cap}(B_r \setminus D) > 0$, then $\tilde{D} = D \cup B_r$ is the desired domain. Hence $\text{cap}(B_{r(p)} \setminus D) = 0$ for every extremal domain D .

In Theorem 5.10, we shall show the existence of extremal domains. We do not succeed yet to give a proof in the case $d = 2$. For the uniqueness, we discuss the case $d = 1$. Finally, we shall comment about open problems for extremal domains.

First we show

PROPOSITION 5.1. *If $c(q) = 1$, then the extremal domain is equal to the unit ball except a set of capacity zero for every $p < q$.*

PROOF. Let D be an extremal domain for $p < q$. Then, by (1) of Lemma 4.1, $h^{(p)}(x, D)^{q/p} \leq h^{(q)}(x, D)$ in D and, by the assumption,

$$1 = c(p) = h^{(p)}(0, D)^{q/p} \leq h^{(q)}(0, D) \leq c(q) \leq 1.$$

Since $h^{(q)}(x, D)$ is harmonic and $h^{(p)}(x, D)^{q/p}$ is subharmonic, $h^{(p)}(0, D)^{q/p} = h^{(q)}(0, D)$ implies $h^{(p)}(x, D)^{q/p} = h^{(q)}(x, D)$ in D and so $(h^{(p)})^{q/p}$ is harmonic in D . From a calculation of $\Delta(h^{(p)})^{q/p}$ which was given in the proof of Lemma 4.1, we see that $h^{(p)}$ is constant and it is equal to 1. Hence $|x|^p \leq h^{(p)}(x, D) = 1$ in D and so $D \subset B_1$. Q.E.D.

We shall introduce a modification of a domain D containing 0 and satisfying $\text{vol } D = \text{vol } B_1$. We call it an outer modification. The new domain depends on ε . We denote it by $M_\varepsilon^0 D$.

Let ε be a number with $0 < \varepsilon < \text{vol } B_1$, let R_ε be a uniquely determined number by $\text{vol}(D \setminus B_{R_\varepsilon}) = \varepsilon$ and let λ_ε be a number greater than one satisfying $\text{vol}(\lambda_\varepsilon D_{R_\varepsilon}) = \text{vol } B_1$, where D_{R_ε} denotes a connected component of $B_{R_\varepsilon} \cap D$ containing 0, note that D_{R_ε} is not $B_{R_\varepsilon} \cap D$ as before. Then, by definition, $\lambda_\varepsilon D_{R_\varepsilon}$ contains 0. We call the domain an outer modification of D and denote it by $M_\varepsilon^0 D$.

Before we discuss the property of the modification, we need the following estimation of harmonic measures:

LEMMA 5.2. *Let C be a number with $0 < C < 1$. Then there exists a number $T > 1$ depending on d , p and C such that $\omega_t(0) \leq Ct^{d-p}\theta_t$ for every $t \geq T$ and for every domain D containing 0 and having the same volume as B_1 has, where $\omega_t(0) = \omega(0, (\partial B_t) \cap D, B_t \cap D)$ and $\theta_t = \theta((\partial B_t) \cap D)$. Further, for every $\varepsilon > 0$, there exists a constant C_2 depending on d , C and ε and we can take T so that $T \leq C_2 p^{(d-1)/d+\varepsilon}$ for $p \geq 1$.*

REMARK. Since $\omega_t(0) \leq v_t(0) = \theta_t$, if $p < d$, we can take $T = 1/C^{1/(d-p)}$. We shall discuss this type of estimation in Lemmas 6.1 and 6.2.

PROOF. Let D be a domain containing 0 and satisfying $\text{vol } D = \text{vol } B_1$. Let θ_t^* be the nonincreasing rearrangement of θ_t and let D^* be a domain containing 0 and satisfying $\theta_t^* = \theta((\partial B_t) \cap D^*)$, see before Lemma 3.7. Take $\lambda \geq 1$ so that $\text{vol } \lambda D^* = \text{vol } D$ and set $\varphi_t = \theta((\partial B_t) \cap \lambda D^*) = \theta_t^*/\lambda$. Then

$$\int_0^R F(\theta_t)t^{-1}dt \geq \int_0^R F(\theta_t^*)t^{-1}dt = \int_0^{\lambda R} F(\varphi_t)t^{-1}dt \geq \int_0^R F(\varphi_t)t^{-1}dt$$

and so, by Lemma 3.4,

$$\omega_R(0) \leq \exp - \int_0^R F(\theta_t)t^{-1}dt \leq \exp - \int_0^R F(\varphi_t)t^{-1}dt.$$

By the proof of Proposition 3.8,

$$\int_0^R F(\varphi_t)t^{-1}dt \geq C_4 R^{d/(d-1)} - C_5$$

for $R > C_3$ and so

$$\omega_t(0) \leq \exp C_5 \exp -C_4 t^{d/(d-1)}$$

for $t > C_3$.

Now we apply the Harnack inequality to $v_t(x) = \omega(x, (\partial B_t) \cap D, B_t)$ defined in B_t and obtain

$$v_t(x) \leq 3 \cdot 2^{d-2} v_t(0) = 3 \cdot 2^{d-2} \theta_t$$

on the closure of $B_{t/2}$. Hence

$$\omega_t(x) \leq \omega_{t/2}(x) \sup_{y \in (\partial B_{t/2}) \cap D} \omega_t(y) \leq \omega_{t/2}(x) \cdot 3 \cdot 2^{d-2} \theta_t$$

in $B_{t/2}$ and so

$$\omega_t(0) \leq 3 \cdot 2^{d-2} \omega_{t/2}(0) \theta_t.$$

Combining this with the above estimation of ω_t replacing t by $t/2$, we have

$$\omega_t(0) \leq 3 \cdot 2^{d-2} \exp C_5 \exp(-C_4 2^{-d/(d-1)} t^{d/(d-1)}) \theta_t$$

for $t > 2C_3$. We write the right-hand side of the inequality as $A \exp(-Bt^{d/(d-1)}) \theta_t$, where A and B are positive constants.

Let C be a given constant satisfying $0 < C < 1$. We shall find a condition on t for the following inequality:

$$A \exp(-Bt^{d/(d-1)}) \leq Ct^{d-p}.$$

This is equivalent to

$$(p-d) \log t \leq Bt^{d/(d-1)} + \log(C/A).$$

Now fix all numbers other than t and let t increase. Then the right-hand side increases more rapidly than the left does; the left may decrease. Hence we can find T such that the inequality holds for every $t \geq T$. Next let $t = C_2 p^{(d-1)/d+\varepsilon}$ and fix all numbers other than p . Then

$$(p-d) \{\log C_2 + ((d-1)/d + \varepsilon) \log p\} \leq BC_2^{d/(d-1)} p^{1+\varepsilon(d/(d-1))} + \log(C/A)$$

holds for large p . Hence we can take $T \leq C_2 p^{(d-1)/d+\varepsilon}$ for large p .

Finally, we show that $T > 1$. Assume that $T \leq 1$. Take $t = 1$ and θ_1 with $0 < \theta_1 < 1$. For every $a > 0$, we can find a domain satisfying $0 \in D$ and $\text{vol } D = \text{vol } B_1$ such that $\omega_1(0) > \theta_1 - a$. Hence the inequality in the lemma does not hold. This contradiction implies $T > 1$. Q.E.D.

Now we discuss the outer modification of domains.

LEMMA 5.3. *Let D be a domain containing 0 and satisfying $\text{vol } D = \text{vol } B_1$. Let T be a number given in Lemma 5.2 depending on d , p and C . Suppose that $h^{(p)}(0, D) \geq 1$ and $D \setminus B_T \neq \emptyset$. If we take $M_\varepsilon^0 D$ so that*

$$0 < \varepsilon \leq \varepsilon_0 = (\sigma/p)(1/C - 1)$$

and $R_\varepsilon \geq T$, then $h^{(p)}(0, M_\varepsilon^0 D) > h^{(p)}(0, D)$.

PROOF. We abbreviate $h^{(p)}(x, D)$ by $h(x)$. Take ε so that $0 < \varepsilon \leq \varepsilon_0$ and $R = R_\varepsilon \geq T$. Set $h_R(x) = h^{(p)}(x, D_R)$ and $\bar{u}_{t,R}(x) = \omega(x, (\partial D_R) \setminus (B_t \cup \partial B_t), D_R)$. Then, by Proposition 2.2 and Corollary 3.3,

$$\begin{aligned} h_R(0) &= p \int_0^R t^{p-1} \bar{u}_{t,R}(0) dt \geq p \int_0^R t^{p-1} u_t(0) dt \\ &= h(0) - p \int_R^\infty t^{p-1} u_t(0) dt. \end{aligned}$$

Since $u_t(0) \leq \omega_t(0) \leq C t^{d-p} \theta_t$ and

$$\begin{aligned} \text{vol}(D \setminus B_R) &= \sigma \int_R^\infty t^{d-1} \theta_t dt = \varepsilon, \\ h_R(0) &\geq h(0) - C(p/\sigma)\varepsilon = h(0) - C\varepsilon', \end{aligned}$$

where $\varepsilon' = (p/\sigma)\varepsilon$.

Next we consider an estimation of $\lambda = \lambda_\varepsilon$ from below. Since $D_R \subset B_R \cap D$,

$$\lambda^p \geq \left(\frac{\text{vol } B_1}{\text{vol } B_1 - \varepsilon} \right)^{p/d} > 1 + \frac{p}{d} (\text{vol } B_1)^{-1} \varepsilon = 1 + \frac{p}{\sigma} \varepsilon = 1 + \varepsilon'.$$

Combining these estimations, we have

$$h^{(p)}(0, M_\varepsilon^0 D) = \lambda^p h_R(0) > (1 + \varepsilon')(h(0) - C\varepsilon') = h(0) + (h(0) - C)\varepsilon' - C(\varepsilon')^2.$$

Since $h(0) \geq 1$ and $\varepsilon' = (p/\sigma)\varepsilon \leq (p/\sigma)\varepsilon_0 = 1/C - 1$,

$$\begin{aligned} h^{(p)}(0, M_\varepsilon^0 D) &> h(0) + (1 - C)\varepsilon' - C(\varepsilon')^2 \\ &= h(0) + (1 - C(1 + \varepsilon'))\varepsilon' \geq h(0). \quad \text{Q.E.D.} \end{aligned}$$

LEMMA 5.4. *Let D be a domain as in Lemma 5.3. Then, for every $a > 0$, we can find $\varepsilon > 0$ such that $M_\varepsilon^0 D \subset B_{T+a}$ and $h^{(p)}(0, M_\varepsilon^0 D) > h^{(p)}(0, D)$.*

PROOF. Let ε_0 be a number as in Lemma 5.3 and let ε_j , $j = 1, \dots, n$, be numbers satisfying $0 < \varepsilon_j \leq \varepsilon_0$ and $R_{\varepsilon_j} \geq T$. We write M_j for $M_{\varepsilon_j}^0$. Then $M_n \cdots M_1 D$ can be written as $M_{\varepsilon(n)}^0 D$ for some $\varepsilon(n) > 0$. We write $R_{(n)}$ and $\lambda_{(n)}$ for $R_{\varepsilon(n)}$ and $\lambda_{\varepsilon(n)}$, respectively. By definition, $R_{(n)}$ decreases as n increases. If $B_\rho \subset D$, then $\rho < R_{(n)}$, because $\rho \geq R_{(n)}$ implies $M_{\varepsilon(n)}^0 D = B_1$ and contradicts $h^{(p)}(0, M_{\varepsilon(n)}^0 D) = h^{(p)}(0, M_n \cdots M_1 D) > h^{(p)}(0, D) \geq 1$. Hence $R_{(n)}$ converges to

a positive number as n increases. This implies that $\lambda_{(n)}$ increases and converges to a positive number, and so λ_{ε_n} tends to 1 as n increases to ∞ . Since $\lambda_{\varepsilon_n}^p \geq 1 + (p/\sigma)\varepsilon_n$, ε_n tends to 0 as n tends to ∞ .

Now we take $\{\varepsilon_j\}_{j=1}^\infty$ as follows: $\varepsilon_1 = \varepsilon_0$ if $v_0 = \text{vol}(D \setminus B_T) \geq \varepsilon_0$, $\varepsilon_1 = v_0$ if $0 < v_0 < \varepsilon_0$, and $\varepsilon_j = \varepsilon_0$ if $v_{j-1} = \text{vol}(M_{j-1} \cdots M_1 D \setminus B_T) \geq \varepsilon_0$, $\varepsilon_j = v_{j-1}$ if $0 < v_{j-1} < \varepsilon_0$ for $j = 2, 3, \dots$, note that $v_j > 0$ for every j . Because ε_n tends to 0 as n tends to ∞ , $\varepsilon_n = v_{n-1}$ for large n and this means that $R_{\varepsilon_n} = T$ and $M_{\varepsilon(n)}^0 D = M_n \cdots M_1 D \subset B_{\lambda_n T}$, where $\lambda_n = \lambda_{\varepsilon_n}$. Since λ_n tends to 1 as n tends to ∞ , we can choose n so that $\lambda_n T \leq T + a$ and $\varepsilon = \varepsilon(n)$ is the desired number. Q.E.D

To make everything clear, we introduce a number $T(p) = T(p, d)$. Let T be a number given in Lemma 5.2 depending on d , p and C . Let $T_C(p)$ be the infimum of such T . $T_C(p)$ decreases as C increases. We define $T(p)$ as $\lim_{C \rightarrow 1} T_C(p)$. $T(p) = 1$ if $p < d$, because $T_C(p) \leq 1/C^{1/(d-p)}$ as mentioned in the remark to Lemma 5.2. By using this notation we have

COROLLARY 5.5. *It follows that $R(p) \leq T(p)$ for every p .*

PROOF. Let T be a number given in Lemma 5.2. Let D be a domain satisfying $0 \in D$, $\text{vol } D = \text{vol } B_1$ and $D \setminus B_{T+a} \neq \emptyset$ for $a > 0$. Then $D \setminus B_T \neq \emptyset$ and so if $h^{(p)}(0, D) \geq 1$, by Lemma 5.4, we can find $\tilde{D} = M_\varepsilon^0 D \subset B_{T+a}$ satisfying $h^{(p)}(0, \tilde{D}) > h^{(p)}(0, D)$. If $h^{(p)}(0, D) < 1$, we set $\tilde{D} = B_1 \subset B_{T+a}$. Hence $T + a$ satisfies the condition on R of the definition of $R(p)$ and so $R(p) \leq T + a$. Since a is arbitrary, we have $R(p) \leq T$ and so $R(p) \leq T_C(p)$. Hence $R(p) \leq T(p)$. Q.E.D.

PROPOSITION 5.6. *It follows that*

- (1) $R(p) = 1$ if $p \leq d$.
- (2) $R(p) \geq C_1 p^{(d-1)/d}$ if $p > d$, where C_1 is the same constant as in Theorem 4.2.
- (3) If $p > d$, for every $\varepsilon > 0$, there is a constant C_2 depending on d and ε such that $R(p) \leq C_2 p^{(d-1)/d+\varepsilon}$.

PROOF. (1) follows from Corollary 2.3. Since $c(p) \leq R(p)$, (2) follows from Theorem 4.2. (3) follows from an estimation of T in Lemma 5.2 and Corollary 5.5. Q.E.D.

REMARK. In the case $d = 1$, every domain containing 0 and having length 2 is contained in B_2 , see the proof of Proposition 3.8. Hence $R(p) \leq 2$ for every $p \geq 1$.

Next we shall introduce another modification of a domain D containing 0 and satisfying $\text{vol } D = \text{vol } B_1$. We call it an inner modification. The domain depends on ε . We denote it by $M_\varepsilon^i D$.

Let ε be a positive number, let r_ε be the supremum of numbers t satisfying $\text{vol}(B_t \setminus D) = \varepsilon$ and let λ_ε be a number less than one satisfying $\text{vol}(\lambda_\varepsilon(B_{r_\varepsilon} \cup D)) = \text{vol } B_1$. By definition, $\lambda_\varepsilon(B_{r_\varepsilon} \cup D)$ contains 0. We call it an inner modification of D and denote it by $M_\varepsilon^i D$. In what follows, we set $\rho(D) = \sup\{t > 0; B_t \subset D\}$. By definition, $\rho(M_\varepsilon^i D) = \lambda_\varepsilon r_\varepsilon$.

Before we discuss arbitrary domains, we introduce admissible domains and deal with their modification. If $d = 1$, every bounded domain containing 0 is admissible. We assume $d \geq 2$ and define admissible domains.

Let $a_1 = 0, a_2, \dots, a_{n+1}$ be strictly increasing numbers and let $B^{(1)}, \dots, B^{(n)}$ be open subsets of ∂B_1 such that $B^{(1)} = \partial B_1$ and $B^{(j-1)} \cap B^{(j)} \neq \emptyset$ for $j = 2, \dots, n$. We assume that the boundary of each $B^{(j)}$ in ∂B_1 is smooth if it is not empty and $d \geq 3$, and $(\partial B_1) \setminus B^{(j)}$ is of positive length if it is not empty and $d = 2$. Set $E^{(j)} = \{x = ry; r \in (a_j, a_{j+1}), y \in B^{(j)}\}$, $j = 1, \dots, n$, $F^{(1)} = \{0\}$, $F^{(j)} = \{x = ry; r = a_j, y \in B^{(j-1)} \cap B^{(j)}\}$, $j = 2, \dots, n$, and $D = \bigcup_{j=1}^n (E^{(j)} \cup F^{(j)})$. We call a domain D admissible if D can be expressed as above. It is easy to show that any domain can be exhausted by admissible domains and every boundary point of an admissible domain is regular with respect to the Dirichlet problem. We note that if $B^{(2)} \neq \partial B_1$, then $\rho(D) = a_2$.

LEMMA 5.7. *Let D be an admissible domain having the same volume as B_1 has. If $q = 2^{1-d}/p < 1$ and if $q\rho(D)^{-d} > 1 + \alpha$ for some $\alpha > 0$, then there is a positive number ε_0 such that*

$$h^{(p)}(0, M_\varepsilon D) \geq h^{(p)}(0, D)(1 + \alpha(p/\sigma)\varepsilon)$$

for every ε with $0 < \varepsilon \leq \varepsilon_0$.

PROOF. For the sake of simplicity, we write ρ, r and λ for $\rho(D), r_\varepsilon$ and λ_ε , respectively. We set $r = \rho + \delta$, $\delta > 0$. We note that ρ is fixed and r and λ varies as ε varies. Further we abbreviate $h^{(p)}(x, D)$ and $h^{(p)}(x, B_r \cup D)$ by $h(x)$ and $h_r(x)$, respectively.

First we take $\varepsilon_{(1)}$ so small that $r = r_\varepsilon \leq \inf\{a_j; a_j > \rho\}$ for every $\varepsilon \leq \varepsilon_{(1)}$. Further conditions on ε we shall mention later. We extend h by setting $h(x) = |x|^p$ on ∂D . Then h is continuous on the closure of D and so $h_r - h$ is continuous on the closure of B_ρ and nonnegative harmonic in B_ρ . Let $\Gamma = (\partial B_\rho) \setminus D$. Since $h_r \geq r^p$ in $B_r \cup D$ and $h = \rho^p$ on Γ ,

$$\begin{aligned} h_r(0) - h(0) &= \int_{\partial B_\rho} (h_r - h) d\theta \\ &\geq \int_{\Gamma} (h_r - r^p) d\theta + \int_{\Gamma} (r^p - \rho^p) d\theta. \end{aligned}$$

We apply the Poisson formula to a positive harmonic function $w = h_r - r^p$ in B_r and obtain

$$w(x) \geq \frac{r - \rho}{r} \left(\frac{r}{r + \rho} \right)^{d-1} w(0) \geq 2^{1-d} \frac{\delta}{r} w(0)$$

on $\Gamma \subset \partial B_\rho$. Hence

$$h_r(0) - h(0) \geq \theta(\Gamma) \{2^{1-d} \delta r^{-1} (h(0) - r^p) + p\rho^{p-1} \delta\}.$$

Noticing

$$\varepsilon = \sigma \int_\rho^r \theta(\Gamma) t^{d-1} dt \leq \sigma \theta(\Gamma) r^{d-1} \delta$$

and setting $\varepsilon' = (p/\sigma)\varepsilon$, we have

$$h_r(0) - h(0) \geq (qh(0) - qr^p + \rho^p)r^{-d}\varepsilon'.$$

Since

$$\begin{aligned}\lambda^p &= \left(\frac{\text{vol } B_1}{\text{vol } B_1 + \varepsilon} \right)^{p/d} > 1 - \frac{p}{d}(\text{vol } B_1)^{-1}\varepsilon \\ &= 1 - \frac{p}{\sigma}\varepsilon = 1 - \varepsilon',\end{aligned}$$

$$\begin{aligned}h^{(p)}(0, M_\varepsilon^i D) &= \lambda^p h_r(0) \\ &\geq (1 - \varepsilon')\{h(0) + (qh(0) - qr^p + \rho^p)r^{-d}\varepsilon'\} \\ &= h(0)\{1 + (qr^{-d} - 1)\varepsilon'\} + \{(\rho^p - qr^p) - (qh(0) + \rho^p - qr^p)\varepsilon'\}r^{-d}\varepsilon'.\end{aligned}$$

Now we mention how to take ε_0 . First take Q so that $q < Q < 1$ and fix it. Next take $\varepsilon_{(2)}$ so small that $(r/\rho)^p \leq Q/q$ for every $\varepsilon \leq \varepsilon_{(2)}$ and set $\varepsilon_{(3)} = (\sigma/p)(1 - Q)\rho^p(qc(p)^p + 1)^{-1}$. Then, for every $\varepsilon \leq \min\{\varepsilon_{(1)}, \varepsilon_{(2)}, \varepsilon_{(3)}\}$,

$$\begin{aligned}\varepsilon' &= (p/\sigma)\varepsilon \leq (1 - Q)\rho^p(qc(p)^p + 1)^{-1} \\ &\leq (\rho^p - qr^p)(qh(0) + \rho^p - qr^p)^{-1}.\end{aligned}$$

Hence

$$h^{(p)}(0, M_\varepsilon^i D) \geq h(0)\{1 + (qr^{-d} - 1)\varepsilon'\}.$$

Finally we take $\varepsilon_{(4)}$ so that $qr^{-d} \geq 1 + \alpha$ for every $\varepsilon \leq \varepsilon_{(4)}$ and set $\varepsilon_0 = \min\{\varepsilon_{(1)}, \varepsilon_{(2)}, \varepsilon_{(3)}, \varepsilon_{(4)}\}$. Thus we obtain the required inequality for every $\varepsilon \leq \varepsilon_0$. Q.E.D.

Now we return to an arbitrary domain and show

LEMMA 5.8. *Let D be a domain containing 0 and having the same volume as B_1 has. It follows that*

- (1) *If $D \neq B_1$, then $\rho(M_\varepsilon^i D) > \rho(D)$.*
- (2) *If $q = 2^{1-d}/p < 1$ and if $q(\rho(M_\varepsilon^i D))^{-d} > 1 + \alpha$ for some $\alpha > 0$, then*

$$h^{(p)}(0, M_\varepsilon^i D) \geq h^{(p)}(0, D)(1 + \alpha(p/\sigma)\lambda_\varepsilon^d\varepsilon).$$

PROOF. We use the same abbreviation as in the proof of Lemma 5.7. To prove (1), assume that $\lambda r \leq \rho = \rho(D)$. Then

$$\text{vol } M_\varepsilon^i D = \text{vol } \lambda B_r + \text{vol } \lambda(D \setminus B_r) \leq \text{vol } B_\rho + \text{vol } (D \setminus B_r).$$

If $B_\rho \neq D$, then $(B_r \setminus B_\rho) \cap D = \emptyset$ and so $\text{vol } B_\rho + \text{vol } (D \setminus B_r) < \text{vol } D$. Hence $\text{vol } M_\varepsilon^i D < \text{vol } B_1$. This is a contradiction and so $\rho(M_\varepsilon^i D) = \lambda r > \rho$.

To prove (2), we may assume that D is admissible. Assume that there are positive numbers $\varepsilon_1, \dots, \varepsilon_n$ such that $M_\varepsilon^i D = M_n \cdots M_1 D$ and

$$h^{(p)}(0, M_j \cdots M_1 D) \geq h^{(p)}(0, M_{j-1} \cdots M_1 D)(1 + \alpha(p/\sigma)\varepsilon_j)$$

for $j = 1, \dots, n$, where $M_j = M_{\varepsilon_j}^i$ and $M_0 D = D$. We write r_j and λ_j for r_{ε_j} and λ_{ε_j} , respectively. Set $\rho_0 = \lambda_n \cdots \lambda_1 \rho(D)$ and $\rho_j = \lambda_n \cdots \lambda_{j+1} \rho(M_j \cdots M_1 D)$ for $j = 1, \dots, n$. Since $\rho(M_j \cdots M_1 D) = \lambda_j r_j > \lambda_j \rho(M_{j-1} \cdots M_1 D)$, $\rho_j > \rho_{j-1}$ for $j = 1, \dots, n$.

To estimate the volume of $\lambda_\varepsilon(B_{r_\varepsilon} \setminus D) = B_{\rho(M_\varepsilon^i D)} \setminus \lambda_\varepsilon D$ we note that $\rho(M_\varepsilon^i D) = \rho_n$, $\lambda_\varepsilon D = \lambda_n \cdots \lambda_1 D$ and

$$B_{\rho_n} \setminus \lambda_n \cdots \lambda_1 D = \bigcup_{j=1}^n (B_{\rho_j} \setminus B_{\rho_{j-1}} \setminus \lambda_n \cdots \lambda_1 D).$$

Since

$$\begin{aligned} B_{\rho_j} \setminus B_{\rho_{j-1}} \setminus \lambda_n \cdots \lambda_1 D &= (\lambda_n \cdots \lambda_j)(B_{r_j} \setminus B_{\rho(M_{j-1} \cdots M_1 D)} \setminus \lambda_{j-1} \cdots \lambda_1 D) \\ &= (\lambda_n \cdots \lambda_j)(B_{r_j} \setminus M_{j-1} \cdots M_1 D), \end{aligned}$$

$\lambda_n \cdots \lambda_j < 1$ and $\text{vol}(B_{r_j} \setminus M_{j-1} \cdots M_1 D) = \varepsilon_j$, we obtain

$$\lambda_\varepsilon^d \varepsilon = \text{vol}(B_{\rho(M_\varepsilon^i D)} \setminus \lambda_\varepsilon D) \leq \sum \varepsilon_j.$$

Combining this with

$$\prod \left(1 + \alpha \left(\frac{p}{\sigma} \right) \varepsilon_j \right) \geq 1 + \alpha \left(\frac{p}{\sigma} \right) \sum \varepsilon_j,$$

we have the desired inequality.

Finally we show the existence of $\varepsilon_1, \dots, \varepsilon_n$ mentioned above. It is sufficient to check the condition on ε_0 in Lemma 5.7. First we have taken $\varepsilon_{(1)}$ so that $r \leq \inf\{a_j; a_j > \rho\}$ for $\varepsilon \leq \varepsilon_{(1)}$. Since n and a_{n+1} are finite, this condition on ε is not essential. The next condition is $(r/\rho)^p \leq Q/q$ for every $\varepsilon \leq \varepsilon_{(2)}$. Since Q/q is fixed and, by (1), $\rho(M_j \cdots M_1 D)$ increases as j increases if $1 > (q/(1+\alpha))^{1/d} > \rho(M_{j-1} \cdots M_1 D)$, we can take $\varepsilon_{(2)}$ independent of j . We can take $\varepsilon_{(3)}$ independent of j by the same reason. By the assumption we can take $\varepsilon_{(4)}$ independent of j . Hence, for every $\varepsilon > 0$ satisfying $q(\rho(M_\varepsilon^i D))^{-d} > 1 + \alpha$, we can find $\varepsilon_1, \dots, \varepsilon_n$ mentioned above. Q.E.D.

PROPOSITION 5.9. *It follows that*

- (1) $r(p) = 1$ if $p \leq d$.
- (2) $2^{1/d-1} p^{-1/d} \leq r(p)$ if $p > d$.

PROOF. (1) follows from Corollary 2.3. To prove (2), let D be a domain containing 0 and having the same volume as B_1 has. Assume that $t < 2^{1/d-1} p^{-1/d}$ and $\text{vol}(B_t \setminus D) > 0$. Then $(2^{1-d}/p)t^{-d} > 1$ and so, by Lemma 5.8, we can find $\tilde{D} = M_\varepsilon^i D$ satisfying $B_t \subset B_{\rho(M_\varepsilon^i D)} \subset M_\varepsilon^i D = \tilde{D}$ and $h^{(p)}(0, \tilde{D}) > h^{(p)}(0, D)$. Hence $t \leq r(p)$ and so (2) holds. Q.E.D.

It is plausible that extremal domains exist for every d and p , but we are able to show

THEOREM 5.10. *There exist extremal domains for every p if $d \neq 2$.*

PROOF. First we deal with the case $d \geq 3$. Let $\{D_n\}$ be a sequence of domains containing 0 and having the same volume as B_1 has such that $h^{(p)}(0, D_n)$ tends to $c(p)^p$ as n tends to ∞ . Here we use D_n to express a sequence of domains, so D_n is neither $B_n \cap D$ nor its connected component as before.

By Propositions 5.6 and 5.9, we may assume that $B_r \subset D_n \subset B_R$ for every n and for some fixed positive numbers r and R . We take R so that $D_n \subset B_{R'} \subset B_R$ for some $R' < R$.

Here we apply spherical rearrangements due to Baernstein [Bae] and Baernstein and Taylor [Bae-Ta]. Let D be a bounded domain containing 0. Then $h^{(p)}(x, D) - |x|^p$ is positive and superharmonic in D . It can be expressed by using the Green function G_D in D as

$$-\frac{1}{(d-2)\sigma} \int G_D(y, x) \Delta_y (h^{(p)}(y, D) - |y|^p) dy = \frac{p(p+d-2)}{(d-2)\sigma} \int G_D(y, x) |y|^{p-2} dy.$$

Let D^* be the spherical rearrangement of D . D^* is defined by $(\partial B_t) \cap D^* = tS(\theta_t)$ if $(\partial B_t) \cap D$ is neither empty nor the full sphere and $(\partial B_t) \cap D^* = (\partial B_t) \cap D$ if $(\partial B_t) \cap D$ is empty or the full sphere, where $\theta_t = \theta((\partial B_t) \cap D)$ and $S(\theta_t)$ denotes a spherical cap defined before Lemma 3.4. We note that D^* is not the same domain defined before Lemma 3.7.

According to a very nice result due to Baernstein and Taylor,

$$\int_{(\partial B_t) \cap D} G_D(y, 0) ds(y) \leq \int_{(\partial B_t) \cap D^*} G_{D^*}(y, 0) ds(y);$$

see [Bae-Ta, p. 267, Corollary 1]. Hence

$$\int_D G_D(y, 0) |y|^{p-2} dy \leq \int_{D^*} G_{D^*}(y, 0) |y|^{p-2} dy$$

and so $h^{(p)}(0, D) \leq h^{(p)}(0, D^*)$. By using this remarkable fact, we may assume that each D_n is invariant under the spherical rearrangement, namely, $D_n^* = D_n$.

Further we may also assume that each D_n has a smooth boundary.

Now we consider functions $h^{(p)}(x, D_n)$ not in D_n but in B_R . We extend each $h^{(p)}(x, D_n)$ by setting $h^{(p)}(x, D_n) = |x|^p$ on $B_R \setminus D_n$. Then $R^p - h^{(p)}(x, D_n)$ is nonnegative, bounded and superharmonic in B_R and it is a Green potential $G\mu_n$ in B_R of a positive measure μ_n . Note that the Green potential is considered in B_R this time and not in D as before. Since

$$\int G\mu_n d\mu_n = \frac{1}{(d-2)\sigma} \int_{B_R} |\text{grad } G\mu_n|^2 dx$$

and

$$\int_{B_R} |\text{grad } G\mu_n|^2 dx \leq \int_{B_R} |\text{grad}(R^p - |x|^p)|^2 dx = \int_{B_R} |\text{grad } |x|^p|^2 dx$$

by the Dirichlet principle, the Green energy $\int G\mu_n d\mu_n$ are uniformly bounded. Hence we can choose a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that $\{\mu_{n_k}\}$ converges weakly to a measure μ with finite Green energy. In what follows we may assume that $\{\mu_n\}$ converges weakly to μ .

We set $D = B_R \setminus \text{supp } \mu$ and we shall show that D is an extremal domain. To do so, it is sufficient to show that

(1) D is an open set containing 0 and has the volume not greater than that of B_1 ,

(2) $\lim_{n \rightarrow \infty} h^{(p)}(0, D_n) = c(p)^p \leq h^{(p)}(0, D)$.

To show (1), we note that $\{\mu_n\}$ also converges vaguely to μ in B_R . Let $\rho < r$ and let f be a nonnegative continuous function in B_R such that $f = 1$ in B_ρ and $\text{supp } f \subset B_r$. Then $\mu(B_\rho) \leq \mu(f) = \lim \mu_n(f) = 0$. Hence $0 \in B_\rho \subset D$. Next let K be a compact subset of D and let f be a nonnegative continuous function in B_R such that $f = 1$ on K and $\text{supp } f \subset D$. Since

$$d\mu_n = -((d-2)\sigma)^{-1}(\Delta G\mu_n)dx = p(p+d-2)((d-2)\sigma)^{-1}|x|^{p-2}dx$$

in $B_R \setminus D_n \setminus \partial D_n$, there is a positive constant a such that $d\mu_n \geq adx$ in $B_R \setminus D_n \setminus \partial D_n$ for every n . Hence

$$\text{vol } K = \text{vol}(K \cap D_n) + \text{vol}(K \setminus D_n) \leq \text{vol } D_n + (1/a)\mu_n(f).$$

Since $\text{vol } D_n = \text{vol } B_1$ and $\mu_n(f) \rightarrow \mu(f) = 0$, $\text{vol } K \leq \text{vol } B_1$ for every compact subset K of D and so $\text{vol } D \leq \text{vol } B_1$.

To prove (2), we set $h(x) = R^p - G\mu(x)$. Since $G\mu(x) \leq \liminf G\mu_n(x)$ in B_R , $\lim h^{(p)}(0, D_n) = \limsup h^{(p)}(0, D_n) \leq h(0)$. Hence it is sufficient to show that $h(x) = h^{(p)}(x, D)$ in D . Since h is harmonic in D and $|x|^p \leq \limsup h^{(p)}(x, D_n) \leq h(x)$ in D , $h^{(p)}(x, D) \leq h(x)$ in D . To show the opposite inequality, let $L = \{(x_1, 0, \dots, 0); -R < x_1 < -r\}$. We shall show that $h(x) \leq |x|^p$ on $(\partial D) \setminus L$.

Let $x_0 \in (\partial D) \setminus L$ and fix it. Let $B_t(x_0) = \{x; |x - x_0| < t\}$. By using the same argument as in the proof of (1), we see that if $B_t(x_0) \subset D_n$ for every n , then $B_\rho(x_0) \subset D$ for every $\rho < t$ and so $B_t(x_0) \subset D$. Hence there exists a subsequence $\{n_k\}$ of $\{n\}$ and $x_{n_k} \in \partial D_{n_k}$ such that $x_{n_k} \rightarrow x_0$ ($k \rightarrow \infty$). Since $D_{n_k}^* = D_{n_k}$ and $x_0 \in (\partial D) \setminus L$, for every $\varepsilon > 0$, we can find $\delta > 0$ such that

$$h^{(p)}(x_0, D_{n_k} \cup B_\delta(x_0)) \leq |x_0|^p + \varepsilon$$

for every $k \geq k_0$. Fix such δ . By using the same argument as above, replacing $G\mu_n(x)$ by $G\mu'_{n_k}(x) = R^p - h^{(p)}(x, D_{n_k} \cup B_\delta(x_0))$, we see that there is a subsequence, say $\{\mu'_{(j)}\}$, of $\{\mu'_{n_k}\}$ and a measure μ' with finite Green energy such that $\mu'_{(j)}$ converges weakly to μ' . Since $G\mu'_{(j)} \leq G\mu_{(j)}$ and $\mu_{(j)}$ (resp. $\mu'_{(j)}$) converges weakly to μ (resp. μ'), $G\mu' \leq G\mu$. Hence $h(x) = R^p - G\mu(x) \leq R^p - G\mu'(x)$. Since $B_\delta(x_0) \subset D_{(j)} \cup B_\delta(x_0)$ and $\mu_{(j)}$ converges vaguely to μ' , $\lim G\mu'_{(j)}(x_0) = G\mu'(x_0)$. Therefore

$$h(x_0) \leq \lim(R^p - G\mu'_{(j)}(x_0)) = \lim h^{(p)}(x_0, D_{(j)} \cup B_\delta(x_0)) \leq |x_0|^p + \varepsilon$$

for every $\varepsilon > 0$. Thus we have proved $h(x) \leq |x|^p$ on $(\partial D) \setminus L$.

Since $d \geq 3$, L is a set of capacity zero and so the inequality holds on ∂D except a set of capacity zero. Hence $h(x) \leq h^{(p)}(x, D)$ in D . Combining this with $h^{(p)}(x, D) \leq h(x)$, we obtain the required equality $h(x) = h^{(p)}(x, D)$ in D .

Next we deal with the case $d = 1$. In this case, a domain containing 0 and having the same volume as B_1 has is just an open interval containing 0 and having length 2. We write $B(\varepsilon) = (-1 + \varepsilon, 1 + \varepsilon)$, where $-1 < \varepsilon < 1$, and let $f(\varepsilon) = h^{(p)}(0, B(\varepsilon))$. Then f is continuous in $(-1, 1)$, $f(0) = 1$ and $\lim_{\varepsilon \rightarrow \pm 1} f(\varepsilon) = 0$, because $h^{(p)}(x, B(\varepsilon))$ is a linear function, say $a(\varepsilon)x + f(\varepsilon)$ in $B(\varepsilon)$ satisfying $|a(\varepsilon)| \leq 2^p/2 = 2^{p-1}$. Hence $f(\varepsilon)$ attains its maximum at some $\varepsilon_M \in (-1, 1)$ and $B(\varepsilon_M)$ is an extremal domain. Q.E.D.

REMARK. The existence theorem holds not only for t^p but also for more general $\Phi(t)$.

We discuss further the case $d = 1$.

PROPOSITION 5.11. *In the case $d = 1$, there is a unique extremal domain $(-1, 1)$ if $1 \leq p \leq 3$ and there are two extremal domains if $p > 3$.*

PROOF. We use the notations at the end of the proof of Theorem 5.10. Let $h(x) = h^{(p)}(x, B(\varepsilon))$. Since $h(-1 + \varepsilon) = (1 - \varepsilon)^p$, $h(1 + \varepsilon) = (1 + \varepsilon)^p$ and h is linear in $(-1, 1)$,

$$f(\varepsilon) = h(0) = \left(\frac{1}{2}\right)\{(1 - \varepsilon)(1 + \varepsilon)^p + (1 + \varepsilon)(1 - \varepsilon)^p\}.$$

If $p = 3$, then $f(\varepsilon) = 1 - \varepsilon^4$. By (1) of Lemma 4.1, $f(\varepsilon) \leq (1 - \varepsilon^4)^{p/3}$ for p with $1 \leq p \leq 3$. Since $f(0) = 1$ for every p , $B(0) = (-1, 1)$ is a unique extremal domain for p with $1 \leq p \leq 3$.

To discuss the case $p > 3$, we note

$$f(\varepsilon) = 1 + (p(p-3)/2)\varepsilon^2 + o(\varepsilon^2)$$

for ε in a neighborhood of 0. Hence $\max f(\varepsilon) > 1$ if $p > 3$. Since $\lim_{\varepsilon \rightarrow \pm 1} f(\varepsilon) = 0$, $f'(\varepsilon_M) = 0$ for some ε_M where $f(\varepsilon)$ attains its maximum. Since $f(0) = 1$ and $f(-\varepsilon) = f(\varepsilon)$, it is sufficient to show that $f'(\varepsilon) = 0$ has at most one solution in $(0, 1)$. By the definition of $f(\varepsilon)$,

$$f'(\varepsilon) = \left(\frac{1}{2}\right)\{(-(1+\varepsilon) + p(1-\varepsilon))(1+\varepsilon)^{p-1} + ((1-\varepsilon) - p(1+\varepsilon))(1-\varepsilon)^{p-1}\}$$

and so $f'(\varepsilon) = 0$ is equivalent to

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{p-1} ((p-1) + \varepsilon(p+1)) - ((p-1) - \varepsilon(p+1)) = 0.$$

Setting $z = (1-\varepsilon)/(1+\varepsilon)$ and $q = p-1$, we have

$$\{(q+1) - z\}z^q - (q+1)z + 1 = 0.$$

We write the left-hand side of the equation as $g(z)$. Then g is a function defined on $[0, 1]$, $g(0) = 1$, $g(1) = 0$ and

$$g''(z) = q(q+1)z^{q-2}\{(q-1) - z\}.$$

Since $q > 2$, $g''(z) > 0$ on $(0, 1]$. Hence g is strictly convex on $(0, 1]$ and $g(z) = 0$ has at most one solution on it other than $z = 1$. Q.E.D.

REMARK. Two extremal domains are $B(\varepsilon_M)$ and $B(-\varepsilon_M) = -B(\varepsilon_M)$. If we regard these domains as the same, Proposition 5.11 asserts the uniqueness of extremal domains in the case $d = 1$. If we consider more general $\Phi(t)$ for t^p , the uniqueness does not hold. Indeed, take $p > 3$ and fix it. Let $\Phi_\delta(t)$ be a continuous function on $[0, \infty)$ which is equal to t^p outside $(1-\delta, 1+\delta)$ and is equal to a linear function $at + b$ in $(1-\delta, 1+\delta)$, where a and b depend on δ , δ is a number with $0 < \delta < \varepsilon_M$ and ε_M is a positive number such that $B(\varepsilon_M)$ is an extremal domain for t^p . Let h be the least harmonic majorant of $\Phi_\delta(|x|)$ in $B(\varepsilon)$ and let $f(\varepsilon) = h(0)$. Then

$$f(\varepsilon) = \left(\frac{1}{2}\right)\{(1-\varepsilon)\Phi_\delta(1+\varepsilon) + (1+\varepsilon)\Phi_\delta(1-\varepsilon)\}$$

and so $f(\varepsilon) = (a+b) - a\varepsilon^2$ on $[-\delta, \delta]$. Hence $f(0) > f(\varepsilon)$ on $[-\delta, \delta]$. Now let us vary δ . If δ increases from 0 to ε_M , then $f(0)$ increases from 1 to $a+b > (a+b) - a\varepsilon_M^2 = h^{(p)}(0, B(\varepsilon_M))$, where a and b are coefficients corresponding to Φ_{ε_M} . Hence we can find $\delta > 0$ such that $f(0) = h^{(p)}(0, B(\varepsilon_M))$. If we take such a δ , $f(\varepsilon)$ attains its maximum at three points; $-\varepsilon_M$, 0 and ε_M . By modifying Φ_δ at $t = 1-\delta$ and $t = 1+\delta$, we have an example of smooth Φ as well.

Finally we summarize open problems on extremal domains. By the definition of $R(p)$, we see that the union of all extremal domains is $B_{R(p)}$ if they exist, because the image of an extremal domain under a rotation around the origin is again extremal. How about for the intersection of extremal domains? In this case, we take the intersection not for all extremal domains, but for all extremal domains maximal for capacity. We say a domain D is maximal for capacity if, for every x on the boundary of D and for every $r > 0$, $\text{cap}(B_r(x) \cap \partial D) > 0$. Is the intersection $B_{r(p)}$? We do not know whether it is open or connected. We do not succeed to

give an estimation of $r(p)$ from above. It will be $r(p) \leq Cp^{-1/d}$ for some constant $C > 0$.

A closely related problem is the following: Is $\theta_t = \theta((\partial B_t) \cap D)$ nonincreasing for every extremal domain? A natural conjecture on extremal domains is the following uniqueness theorem: An extremal domain maximal for capacity is determined uniquely except rotations around the origin. If this conjecture is true, $r(p)$ and $R(p)$ are inner and outer radius of an extremal domain maximal for capacity, respectively.

An interesting problem is to give an explicit equation defining the boundary of an extremal domain maximal for capacity. For example, if $p \leq d$, then B_1 is an extremal domain maximal for capacity and its boundary is given by $x_1^2 + \cdots + x_d^2 = 1$. In the general case, the problem seems to be very difficult.

6. Further estimation of $c(p)$. In Theorem 1.4 we proved that $c(p) = 1$ if $p \leq d$ and we gave other estimations of $c(p)$ in Theorem 4.2. Are there any $p > d$ satisfying $c(p) = 1$? If $d = 1$ we have seen in the proof of Proposition 5.11 that $c(p) = 1$ for p with $1 \leq p \leq 3$. Hence such p exist for $d = 1$. In this section we shall show such p do exist for every $d \geq 2$.

First we introduce a number $T_1(p)$ and discuss the relation between $T_1(p)$ and $T(p)$ defined before Corollary 5.5. We define $T_1(p)$ as the infimum of $T \geq 1$ satisfying $\omega_t(0) \leq t^{d-p}\theta_t$ for every $t \geq T$ and for every domain D containing 0 and having the same volume as B_1 has, where $\omega_t(0) = \omega(0, (\partial B_t) \cap D, B_t \cap D)$ and $\theta_t = \theta((\partial B_t) \cap D)$.

LEMMA 6.1. *It follows that*

- (1) *Both $T(p)$ and $T_1(p)$ are nondecreasing functions of p .*
- (2) *$1 \leq T_1(p) \leq T(p)$ for every p .*
- (3) *For every p and $\varepsilon > 0$, $T(p - \varepsilon) \leq T_1(p)$.*
- (4) *$c(p) \leq T_1(p)$ for every p .*

PROOF. By the definition of $T(p)$ and $T_1(p)$, (1) and (2) follow immediately. Let $a > 0$ and $t \geq T_1(p) + a$. Since

$$\frac{1}{t^p} = \frac{1}{t^\varepsilon} \frac{1}{t^{p-\varepsilon}} \leq \frac{1}{(1+a)^\varepsilon} \frac{1}{t^{p-\varepsilon}}$$

for every $t \geq T_1(p) + a \geq 1 + a$, by the definition of $T_C(p)$,

$$T_{1/(1+a)^\varepsilon}(p - \varepsilon) \leq T_1(p) + a.$$

Hence $T(p - \varepsilon) \leq T_1(p) + a$ for every $a > 0$ and so (3) holds.

If $d = p = 1$, then (4) holds because $c(p) = 1$. In other cases (4) follows from (3), because $c(p - \varepsilon) \leq R(p - \varepsilon) \leq T(p - \varepsilon) \leq T_1(p)$ and $c(p)$ is continuous. Q.E.D.

There is another characterization of $T_1(p)$. Let D be a domain containing 0. We do not assume that $\text{vol } D = \text{vol } B_1$. Let D_t be a connected component of $B_t \cap D$ containing 0 and let r_t be the volume radius $r(D_t)$ of D_t .

LEMMA 6.2. *It follows that*

$$\omega_t(0) \leq (r_t/t)^{p-d}\theta_t$$

for every t satisfying $r_t/t \leq 1/T_1(p)$. In particular, the inequality holds for every $t > 0$ if $T_1(p) = 1$. Furthermore, $\omega_1(0) \leq r_1^{p-d}\theta_1$ for every domain D satisfying $r_1 = r(D_1) \leq 1/T_1(p)$.

PROOF. Take $\lambda > 0$ so that $\text{vol}(\lambda D_t) = \text{vol } B_1$. Then $\lambda = 1/r_t$. If $r_t/t \leq 1/T_1(p)$, then $\lambda t \geq T_1(p)$ and so $\omega'_{\lambda t}(0) \leq (\lambda t)^{d-p}\theta'_{\lambda t}$, where ω' and θ' denote ω and θ corresponding to λD_t , respectively. Since $\omega'_{\lambda t}(0) = \omega_t(0)$ and $\theta'_{\lambda t} = \theta_t$, we have the desired inequality. It follows that $r_t/t \leq 1$ for every $t > 0$, because $r_t = r(D_t) \leq r(B_t) = t$. Hence if $T_1(p) = 1$, the inequality holds for every $t > 0$. Q.E.D.

Next we show

PROPOSITION 6.3. *It follows that $T_1(d + 2^{1-d}) = 1$.*

PROOF. We shall show

$$\omega_t(0) \leq (r_t/t)^{2^{1-d}} \theta_t$$

for every $t > 0$ and for every domain D containing 0. We may assume that D is an admissible domain defined before Lemma 5.7.

Take $t > 0$ and fix it. For s with $0 < s \leq t$, we set

$$w_s(x) = \omega(x, (\partial B_t) \cap D, B_s \cup D_t).$$

Then $w_s = \omega_t$ for sufficiently small s and w_s increases and converges to v_t as s increases and tends to t .

Now we use the same argument as in the proof of Lemma 5.7 replacing D , ρ , r , h and h_r by D_t , s , $s + \delta$, w_s and $w_{s+\delta}$, respectively. Then, for sufficiently small δ , we have

$$\begin{aligned} w_{s+\delta}(0) - w_s(0) &= \int_{\partial B_s} (w_{s+\delta} - w_s) d\theta \geq \int_{(\partial B_s) \setminus D_t} w_{s+\delta} d\theta \\ &\geq 2^{1-d} \sigma^{-1} (s + \delta)^{-d} \varepsilon w_{s+\delta}(0), \end{aligned}$$

where $\varepsilon = \sigma \int_s^{s+\delta} \theta((\partial B_s) \setminus D_t) \tau^{d-1} d\tau$. Taking an increasing sequence $\{s_j\}_{j=0}^k$ of positive numbers so that $w_{s_0} = \omega_t$, $s_k = t$ and $s_j - s_{j-1}$ is sufficiently small for each $j = 1, \dots, k$, we have

$$\begin{aligned} \log \left(\frac{\theta_t}{\omega_t(0)} \right) &= \sum_{j=1}^k \log \left(\frac{w_{s_j}(0)}{w_{s_{j-1}}(0)} \right) \\ &\geq \sum_{j=1}^k \frac{w_{s_j}(0) - w_{s_{j-1}}(0)}{w_{s_j}(0)}. \end{aligned}$$

Combining these inequalities and letting $k \rightarrow \infty$, we have

$$\log \left(\frac{\theta_t}{\omega_t(0)} \right) \geq 2^{1-d} \sigma^{-1} \int_{B_t \setminus D_t} |x|^{-d} dx.$$

Let $A = \{x; a < |x| < t\}$ and take a so that $\text{vol } A = \text{vol}(B_t \setminus D_t)$. Then

$$\int_{B_t \setminus D_t} |x|^{-d} dx \geq \int_A |x|^{-d} dx = \sigma \left(\log \left(\frac{t}{a} \right) \right)$$

and $a = r_t$. Hence

$$\log \left(\frac{\theta_t}{\omega_t(0)} \right) \geq 2^{1-d} \log \left(\frac{t}{r_t} \right). \quad \text{Q.E.D.}$$

Combining (4) of Lemma 6.1 with Proposition 6.3, we have the following main theorem:

THEOREM 6.4. *It follows that $c(p) = 1$ if $p \leq d + 2^{1-d}$.*

7. A conjecture and supporting facts. It is natural to ask “What is the greatest value of p satisfying $c(p) = 1$?” Our conjecture is

CONJECTURE. The greatest value of p satisfying $c(p) = 1$ is $d + 2$, namely, $c(p) = 1$ for $p \leq d + 2$.

A stronger conjecture is $r(p) = R(p) = 1$ for $p \leq d + 2$. By (3) of Lemma 6.1 and Proposition 6.3, $R(p) = 1$ if $p < d + 2^{1-d}$. We note that if $d = 1$, then the conjecture is true. We write this as the following proposition. The proof is omitted because it was given in the proof of Proposition 5.11.

PROPOSITION 7.1. *Assume that $d = 1$. Then $c(p) = 1$ for p with $1 \leq p \leq 3$ and $c(p) > 1$ for $p > 3$.*

To show the next supporting fact, let us consider domains close to the unit ball. For a continuously differentiable function ρ defined on the unit sphere ∂B_1 and satisfying $\int_{\partial B_1} \rho ds = 0$, take a family $\{\rho_\varepsilon\}_{\varepsilon>0}$ of continuously differentiable functions ρ_ε defined on ∂B_1 for every small $\varepsilon > 0$ such that

(1) $\text{vol } \Omega_\varepsilon = \text{vol } B_1$, where $\Omega_\varepsilon = \{x = |x|y; |x| < 1 + \varepsilon\rho(y), y \in \partial B_1\}$,

(2) $\int_{\partial B_1} \rho_\varepsilon^2 ds \rightarrow \int_{\partial B_1} \rho^2 ds$ ($\varepsilon \rightarrow 0$),

(3) $\int_{\Omega_\varepsilon} |\text{grad } H_\varepsilon^{\rho_\varepsilon}|^2 dx \rightarrow \int_{B_1} |\text{grad } H^\rho|^2 dx$ ($\varepsilon \rightarrow 0$),

where $H_\varepsilon^{\rho_\varepsilon}(x) = H^{\rho_\varepsilon(x/|x|)}(x, \Omega_\varepsilon)$ and $H^\rho(x) = H^\rho(x, B_1)$.

For every ρ , there exists a family $\{\rho_\varepsilon\}$ satisfying (1) to (3). For example, let $V_\varepsilon = \{x = |x|y; |x| < 1 + \varepsilon\rho(y), y \in \partial B_1\}$ and set $\Omega_\varepsilon = r(V_\varepsilon)^{-1}V_\varepsilon$. Then $1 + \varepsilon\rho(y) = r(V_\varepsilon)^{-1}(1 + \varepsilon\rho(y))$ and so ρ_ε is continuously differentiable on ∂B_1 . By definition, (1) is satisfied. Since $\int_{\partial B_1} \rho ds = 0$,

$$\text{vol } V_\varepsilon = \frac{1}{d} \int_{\partial B_1} (1 + \varepsilon\rho)^d ds = \frac{\sigma}{d} (1 + o(\varepsilon))$$

and so $r(V_\varepsilon)^{-1} = 1 + o(\varepsilon)$. Hence ρ_ε converges uniformly to ρ and (2) follows. By using the same argument as above we see that $\partial\rho_\varepsilon/\partial y_j$ converges uniformly to $\partial\rho/\partial y_j$ on ∂B_1 . Hence

$$\int_{\Omega_\varepsilon} |\text{grad}(H_\varepsilon^{\rho_\varepsilon} - H_\varepsilon^\rho)|^2 dx \rightarrow 0 \quad (\varepsilon \rightarrow 0),$$

where $H_\varepsilon^\rho(x) = H^\rho(x/|x|)(x, \Omega_\varepsilon)$ and

$$\int_K |\text{grad}(H_\varepsilon^\rho - H^\rho)|^2 dx \rightarrow 0 \quad (\varepsilon \rightarrow 0)$$

for every compact subset K of B_1 . Thus (3) is satisfied.

We say Ω_ε is close to B_1 if ε is small. We shall show

PROPOSITION 7.2. *If $p < d + 2$ and $\rho \neq 0$, then $h^{(p)}(0, \Omega_\varepsilon) < 1$ for every Ω_ε sufficiently close to B_1 . The same holds if $p = d + 2$ and ρ is not a linear function of y_1, \dots, y_d .*

To prove the proposition, we need the following lemma:

LEMMA 7.3. *Let ρ be a continuously differentiable function defined on $B_1 \cup \partial B_1$ satisfying $\int_{\partial B_1} \rho \, ds = 0$. Then*

$$\int_{\partial B_1} \rho^2 \, ds \leq \int_{B_1} |\operatorname{grad} \rho|^2 \, dx.$$

Equality holds if and only if ρ is a linear function of x_1, \dots, x_d .

PROOF. Let $H^\rho(x) = H^\rho(x, B_1)$ and write $D_\Omega(f)$ for $\int_\Omega |\operatorname{grad} f|^2 \, dx$. Then, by the Dirichlet principle, $D_{B_1}(H^\rho) \leq D_{B_1}(\rho)$. Hence we may assume that ρ is harmonic in B_1 . Let $\rho(x) = \sum_{j=1}^\infty H_j(x)$ be an expansion of ρ in B_1 by spherical functions, where H_j denotes a spherical function of degree j .

Since $\partial H_j / \partial n_y = j H_j(y) / r$ on ∂B_r , where $\partial / \partial n_y$ denotes the exterior normal derivative at $y \in \partial B_r$, by the Green formula,

$$\begin{aligned} D_{B_r}(\rho) &= \int_{\partial B_r} \rho \frac{\partial \rho}{\partial n} \, ds - \int_{B_r} \rho \Delta \rho \, dx \\ &= \frac{1}{r} \int_{\partial B_r} \left(\sum H_j \right) \left(\sum j H_j \right) \, ds. \end{aligned}$$

Noticing $\int_{B_r} H_j H_k \, ds = 0$ if $j \neq k$, we have

$$D_{B_r}(\rho) = \frac{1}{r} \sum j \int_{\partial B_r} H_j^2 \, ds.$$

Since

$$\frac{1}{r} \int_{\partial B_r} \rho^2 \, ds = \frac{1}{r} \int_{\partial B_r} \left(\sum H_j \right)^2 \, ds = \frac{1}{r} \sum \int_{\partial B_r} H_j^2 \, ds,$$

by letting $r \rightarrow 1$, we have the required inequality. Equality holds if and only if $H_j = 0$ for every $j \geq 2$, namely, ρ is a linear function H_1 of x_1, \dots, x_d . Q.E.D.

PROOF OF PROPOSITION 7.2. We may assume that $d \geq 2$, because we have treated the case $d = 1$ in the proof of Proposition 5.11.

First we note that

$$(d-2)\sigma u(0) = \int_{\partial \Omega} \frac{1}{r^{d-2}} \frac{\partial u}{\partial n_y}(y) \, ds_y - \int_{\partial \Omega} u(y) \frac{\partial}{\partial n_y} \left(\frac{1}{r^{d-2}} \right) \, ds_y$$

and

$$\int_{\partial \Omega} \frac{\partial u}{\partial n_y}(y) \, ds_y = 0$$

for every u harmonic in Ω and continuously differentiable on the closure of Ω , where Ω is a domain with smooth boundary containing 0, $r = |y|$, $\partial / \partial n_y$ denotes the exterior normal derivative at $y \in \partial \Omega$ and ds_y denotes the surface area element of $\partial \Omega$. These are obtained by the Green formula. In the above and in what follows, if $d = 2$, we replace $(d-2)\sigma$ and $1/r^{d-2}$ by 2π and $\log(1/r)$, respectively. We shall apply these equalities to $u(x) = h^{(p)}(x, \Omega_\varepsilon) = H_\varepsilon^{r^p}(x, \Omega_\varepsilon)$. We write $H_\varepsilon^{r^p}(x)$ for $H^{r^p}(x, \Omega_\varepsilon)$.

To calculate $\int_{\partial\Omega_\varepsilon} r^{2-d} \partial H_\varepsilon^{r^p} / \partial n_y ds_y$, we use the second equality and know that it is equal to

$$\int_{\partial\Omega_\varepsilon} \left(\frac{1}{r^{d-2}} - 1 \right) \frac{\partial H_\varepsilon^{r^p}}{\partial n_y} ds_y.$$

Since $r^{2-d} - 1 = -(d-2)\varepsilon\rho_\varepsilon(y/|y|) + o(\varepsilon)$,

$$\frac{\partial H_\varepsilon^{r^p}}{\partial n_y}(y) = \frac{\partial H^{1+p\varepsilon\rho_\varepsilon(y/|y|)+o(\varepsilon)}(y, \Omega_\varepsilon)}{\partial n_y} = p\varepsilon \frac{\partial H_\varepsilon^{\rho_\varepsilon}}{\partial n_y}(y) + o(\varepsilon)$$

on $\partial\Omega_\varepsilon$ and

$$D_{\Omega_\varepsilon}(H_\varepsilon^{\rho_\varepsilon}) = \int_{\partial\Omega_\varepsilon} \rho_\varepsilon \left(\frac{y}{|y|} \right) \frac{\partial H_\varepsilon^{\rho_\varepsilon}}{\partial n_y} ds_y,$$

the integral is equal to

$$-(d-2)p\varepsilon^2 D_{\Omega_\varepsilon}(H_\varepsilon^{\rho_\varepsilon}) + o(\varepsilon^2).$$

To calculate $-\int_{\partial\Omega_\varepsilon} H_\varepsilon^{r^p} \partial r^{2-d} / \partial n_y ds_y$, we note that

$$-\frac{\partial}{\partial n_y} \left(\frac{1}{r^{d-2}} \right) ds_y = (d-2) \frac{(n_y, y/|y|)}{r^{d-1}} \frac{r^{d-1} \sigma d\theta}{(n_y, y/|y|)} = (d-2) \sigma d\theta.$$

Then it is equal to

$$\begin{aligned} -\int_{\partial\Omega_\varepsilon} r^p \frac{\partial}{\partial n_y} \left(\frac{1}{r^{d-2}} \right) ds_y &= (d-2) \int_{\partial B_1} (1 + \varepsilon\rho_\varepsilon)^p ds \\ &= (d-2) \left\{ \sigma + p\varepsilon \int \rho_\varepsilon ds + \frac{p(p-1)}{2} \varepsilon^2 \int \rho_\varepsilon^2 ds + o(\varepsilon^2) \right\}. \end{aligned}$$

Since

$$\text{vol } \Omega_\varepsilon = \frac{1}{d} \int_{\partial B_1} (1 + \varepsilon\rho_\varepsilon)^d ds = \frac{\sigma}{d}$$

and

$$\int (1 + \varepsilon\rho_\varepsilon)^d ds = \sigma + d\varepsilon \int \rho_\varepsilon ds + \frac{d(d-1)}{2} \varepsilon^2 \int \rho_\varepsilon^2 ds + o(\varepsilon^2),$$

we have

$$\varepsilon \int \rho_\varepsilon ds = -\frac{d-1}{2} \varepsilon^2 \int \rho_\varepsilon^2 ds + o(\varepsilon^2).$$

Hence the integral is equal to

$$(d-2) \left\{ \sigma + \frac{p(p-d)}{2} \varepsilon^2 \int \rho_\varepsilon^2 ds + o(\varepsilon^2) \right\}.$$

Adding these two and dividing by $d-2$, we have

$$\sigma \{h^{(p)}(0, \Omega_\varepsilon) - 1\} = \left\{ -p D_{\Omega_\varepsilon}(H_\varepsilon^{\rho_\varepsilon}) + \frac{p(p-d)}{2} \int_{\partial B_1} \rho_\varepsilon^2 ds \right\} \varepsilon^2 + o(\varepsilon^2).$$

If ρ satisfies the assumption, by Lemma 7.3 and conditions (2) and (3) of $\{\rho_\varepsilon\}$, the right-hand side becomes negative for sufficiently small ε and so the proposition follows. Q.E.D.

Next consider the case $p = d + 2$ and ρ is a linear function of y_1, \dots, y_d . We may assume that $\rho(y) = y_1$. Let us take $\Omega_\varepsilon = B_1(\varepsilon e) = \{x; |x - \varepsilon e| < 1\}$, where $e = (1, 0, \dots, 0)$. Then, by the Poisson formula,

$$h^{(p)}(0, B_1(\varepsilon e)) = \frac{1}{\sigma} \int_{\partial B_1(\varepsilon e)} \frac{1 - \varepsilon^2}{|y|^d} |y|^p ds_y.$$

Set $Y = y - \varepsilon e$. Since $|y|^2 = |Y + \varepsilon e|^2 = 1 + 2Y_1\varepsilon + \varepsilon^2$ and $\int_{\partial B_1} Y_1 ds_Y = 0$,

$$h^{(d+2)}(0, B_1(\varepsilon e)) = 1 - \varepsilon^4.$$

Combining this with Proposition 7.2, we guess that $h^{(d+2)}(0, \Omega_\varepsilon) < 1$ for $\rho \neq 0$ and Ω_ε sufficiently close to B_1 . Finally we show

PROPOSITION 7.4. *If $p > d + 2$, then $c(p) > 1$.*

PROOF. Let $B_1(\varepsilon e)$ and $Y = y - \varepsilon e$ be as above. Since

$$\begin{aligned} |y|^{p-d} &= (|y|^2)^{(p-d)/2} \\ &= 1 + (p-d)Y_1\varepsilon + \left\{ \frac{p-d}{2} + \frac{(p-d)(p-d-2)}{2} Y_1^2 \right\} \varepsilon^2 + o(\varepsilon^2) \end{aligned}$$

and

$$\int_{\partial B_1} Y_1^2 ds_Y = \frac{\sigma}{d},$$

by the Poisson formula written above,

$$h^{(p)}(0, B_1(\varepsilon e)) = 1 + \frac{p(p-d-2)}{2d} \varepsilon^2 + o(\varepsilon^2).$$

Hence the proposition follows. Q.E.D.

8. Applications. In this section, we deal with four applications of our estimations.

1. *Sharp estimation of the Hardy norms by the image areas of functions.* Let U be the unit disk in the complex plane. For $p > 0$, we denote $H^p = H^p(U)$ the class of all holomorphic functions f with finite Hardy norms $\|f\|_{H^p}$, we define them by

$$\|f\|_{H^p} = \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{is})|^p ds \right)^{1/p}.$$

As mentioned in the introduction, Alexander, Taylor and Ullman [**A-Ta-U**] proved that if f is holomorphic in U and vanishes at 0, then $\|f\|_{H^2} \leq ((1/\pi) \text{area } f(U))^{1/2}$. If we use the volume radius $r(f(U))$ of $f(U)$, then we have a simpler form: $\|f\|_{H^2} \leq r(f(U))$. There are at least five proofs. They were given by Alexander, Taylor and Ullman [**A-Ta-U**], Alexander and Osserman [**A-O**], Alexander [**A**], Kobayashi [**Ko**] and Khavinson [**Kh**]. The equality assertion was given by [**A-O**] and [**Ko**]. Kobayashi's proof is very nice, because the equality assertion follows immediately from the proof. Here we use his idea and improve the result.

THEOREM 8.1. *Let f be a holomorphic function in U satisfying $f(0) = 0$. Then*

$$\|f\|_{H^p} \leq c(p) \left(\frac{1}{\pi} \text{area } f(U) \right)^{1/2} = c(p) r(f(U)).$$

If $p \leq 2 + \frac{1}{2}$, then we can take $c(p) = 1$. If $p < 2 + \frac{1}{2}$, equality holds in finite values if and only if f is a constant multiple of an inner function.

PROOF. We may assume that f is not constant and area $f(U)$ is finite. Let $D = f(U)$, $h(x) = h^{(p)}(x, D)$ and $v = h \circ f$. Then v is harmonic and satisfies $|f|^p \leq v$ in U . Hence, by the mean-value property of harmonic functions,

$$\|f\|_{H^p} \leq \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} v(re^{is}) ds \right)^{1/p} = v(0)^{1/p} = h(0)^{1/p}.$$

By the definition of $c(p)$ given at the end of §1, the desired inequality follows. If $p \leq 2 + \frac{1}{2}$, then, by Theorem 6.4, $c(p) = 1$.

Assume that $\|f\|_{H^p} = h(0)^{1/p} = r(f(U)) < \infty$ for $p < 2 + \frac{1}{2}$. By $h(0)^{1/p} = r(f(U))$, we see that $(1/r(f(U)))f(U)$ is an extremal domain for p . We write ρ for $r(f(U))$. Since $c(2 + \frac{1}{2}) = 1$, by Proposition 5.1, $f(U) = B_\rho \setminus E$, where E is a set of capacity zero. Let $f^*(e^{is})$ be the radial limit of the bounded function f . Then $|f^*| \leq \rho$ a.e. on ∂U and

$$\|f\|_{H^p} = \left(\frac{1}{2\pi} \int_0^{2\pi} |f^*(e^{is})|^p ds \right)^{1/p} \leq \rho.$$

Hence $\|f\|_{H^p} = \rho$ implies that $|f^*| = \rho$ a.e. on ∂U and so f/ρ is an inner function.

Conversely, if f/ρ is an inner function, then $f(U) \subset B_\rho$ and $r(f(U)) \leq \rho$. Since $|f^*| = \rho$ a.e. on ∂U , $\|f\|_{H^p} = \rho$ and so $r(f(U)) \leq \rho = \|f\|_{H^p}$. The opposite inequality holds for every f and equality holds for a constant multiple f of an inner function. Q.E.D.

From our proof we see that an inner function takes all values in the unit disk except a set of capacity zero, a result due to O. Frostman.

By using a similar argument to the above, we also have an estimation of the Hardy norms of functions defined on Riemann surfaces. Namely, let R be a Riemann surface having a Green function. Take a point ξ in R and fix it. Let f be a holomorphic function in R and let u be the least harmonic majorant of $|f|^p$ in R . Set $\|f\|_{H^p} = u(\xi)^{1/p}$.

THEOREM 8.2. *Let R , ξ , f and u be as above. If $f(\xi) = 0$, then the inequality replacing U by R in Theorem 8.1 holds. The equality assertion is also valid if a constant multiple of an inner function is replaced by a map of type-BI, in the sense of Heins [Hei], from R into a disk centered at the origin.*

PROOF. It is sufficient to show the equality assertion. Let $h(x) = h^{(p)}(x, f(R))$ and $v = h \circ f$. Then $u \leq v$ in R and

$$\|f\|_{H^p}^p = u(\xi) \leq v(\xi) = h(0) \leq r(f(R))^p.$$

Hence $\|f\|_{H^p} = r(f(R)) < \infty$ if and only if $u(\xi) = v(\xi)$ and $h(0) = r(f(R))^p < \infty$. We have seen in the proof of Theorem 8.1 that $h(0) = r(f(R))^p < \infty$ is equivalent to $f(R) = B_{r(f(R))} \setminus E$, where E is a set of capacity zero.

Since $u \leq v$ in R , $u(\xi) = v(\xi)$ is equivalent to $u = v$. If we denote by \hat{s}_R the least harmonic majorant of a subharmonic function s defined in a Riemann surface R , then $u = v$ can be written as

$$(s \circ f)_R = \hat{s}_{f(R)} \circ f,$$

where $s(x) = |x|^p$ and it is not harmonic in $f(R)$. Therefore, by Theorem 1 in Kobayashi and Suita [Ko-Su], $u = v$ if and only if f is of type-Bl.

Finally if f is a map of type-Bl from R in B_ρ , then $B_\rho \setminus f(R)$ is a set of capacity zero. Thus the proof is complete. Q.E.D.

A map of type-Bl from U into a disk B_ρ is nothing but a constant multiple of an inner function, see [Hei, pp. 453–454]. Therefore the proof of Theorem 8.2 is an alternate proof of equality assertion in Theorem 8.1.

2. *Hardy classes and images of functions.* Let f be a holomorphic function in U . If there exists a harmonic majorant h of $|x|^p$ in $f(U)$, then, by using the same argument as in 1, $f \in H^p(U)$ and $\|f\|_{H^p} \leq h(f(0))^{1/p}$. As a good survey in this direction, we refer to Hansen [Han].

We shall apply our Proposition 3.14 and Corollary 3.15 and construct three examples. The first example concerns that of Hansen [Han, p. 245, Example II]. Let ψ be an increasing continuous function on $[1, \infty)$ with $\psi(1) = 0$ and $\lim_{r \rightarrow \infty} \psi(r)/r = \infty$. Set $D = \mathbf{R}^2 \setminus ([0, e] \cup \{\exp(r + i\psi(r)); r \geq 1\})$. Hansen proved that there exists a harmonic majorant of $|x|^p$ in D for every $p > 0$ by using the Ahlfors distortion theorem. Our example differs from his: Domains are not simply connected.

EXAMPLE 1. For given $p > 0$, take a natural number n and $\lambda > 1$ so that $\lambda^4 = 16^{1/n} \leq 1 + (3\pi/8)2^{-1/2}4^{-p}$. Note that $\lambda^n = 2$. Set

$$E_{jk} = [j \log 4 + (2k - 1) \log \lambda, j \log 4 + 2k \log \lambda]$$

and

$$E = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^n E_{jk}.$$

Let ψ be a continuous function on E satisfying $\max_{E_{jk}} \psi - \min_{E_{jk}} \psi \geq 2\pi$ for every j and k . We do not assume that ψ is increasing. Consider a domain $D = \mathbf{R}^2 \setminus \{\exp(r + i\psi(r)); r \in E\}$ and apply Proposition 3.14 and Corollary 3.15. Since $d = 2$, $C = 3$ and so we shall show $\alpha(t) \geq \frac{2}{9}16^{p-j}$ for $t \in [4^j, 2 \cdot 4^j]$. Set $A = \{x \in \mathbf{R}^2; t < |x| < 2t\}$ and $A_k = \{x; 4^j \lambda^{2(k-1)} \leq |x| < 4^j \lambda^{2k}\}$. Let k_1 and k_2 be numbers satisfying $\bigcup_{k=k_1}^{k_2} A_k \subset A \subset \bigcup_{k=k_1-1}^{k_2+1} A_k$ and set $A'_{k_1} = \{x; t < |x| < 4^j \lambda^{2k_1}\}$ and $A'_{k_2} = \{x; 4^j \lambda^{2(k_2-1)} \leq |x| < 2t\}$. Then $A = A'_{k_1} \cup A'_{k_2} \cup \bigcup_{k=k_1+1}^{k_2-1} A_k$ and for each A_k or A'_k , by (1) of Proposition 3.11,

$$\begin{aligned} \int |\text{grad } f|^2 dx &\geq \frac{1}{\lambda^4} \left(\frac{\pi}{2(4^{j+1} - 4^{j+1}\lambda^{-4})} \right)^2 \int f^2 dx \\ &\geq \left(\frac{\pi}{8} \right)^2 \frac{16^{-j}}{(\lambda^4 - 1)^2} \int f^2 dx \end{aligned}$$

for every continuously differentiable function f in A_k or A'_k which vanishes outside of D . Hence

$$\alpha(t) \geq \left(\frac{\pi}{8} \right)^2 \frac{16^{-j}}{(\lambda^4 - 1)^2} \geq \left(\frac{2}{9} \right) 16^{p-j}$$

and, by Corollary 3.15, $h^{(p)}(0, D) < \infty$.

If p increases, then n increases. If we take n replacing by n_j so that $n_j \rightarrow \infty$ as $j \rightarrow \infty$, we see that there exists a harmonic majorant of $|x|^p$ in the modified domain for every $p > 0$.

The second example is a slight modification of Example 1, but it is interesting because $\partial B_t \subset D$ for every t except a countable set.

EXAMPLE 2. Let n and λ be as in Example 1 and set

$$D = \mathbf{R}^2 \setminus \bigcup_{j=1}^{\infty} \bigcup_{k=1}^n [\{x \in \mathbf{R}^2; |x| = 4^j \lambda^{(2k-1)}, x_1 \geq 0\} \\ \cup \{x \in \mathbf{R}^2; |x| = 4^j \lambda^{2k}, x_1 \leq 0\}].$$

By the same argument as in Example 1, we see that $h^{(p)}(0, D) < \infty$.

The third example is the case such that $\lim_{t \rightarrow \infty} \text{area } B_t \cap D / (\pi t^2) = 0$. It was dealt with by Hansen and Hayman [Han-Hay, Theorem 1] and they proved that there exists a harmonic majorant of $|x|^p$ in D for every $p > 0$ by using the Tsuji inequality. In Example 3, we shall first give an alternate proof of the above result and give another example having a harmonic majorant of $|x|^p$ for every $p > 0$.

EXAMPLE 3. Let D be a domain containing 0 and satisfying $\varepsilon(t) = \text{area } B_t \cap D / (\pi t^2) \rightarrow 0$ ($t \rightarrow \infty$). Let $Q = Q_{nk_1 k_2}$ be cubes having sides of length n defined by $Q = \{x \in \mathbf{R}^2; (k_1 - 1)n \leq x_1 < k_1 n, (k_2 - 1)n \leq x_2 < k_2 n\}$. To apply Corollary 3.15, for each natural number j , let us take $n = n(j)$ the least natural number satisfying $n^2/2 \geq \varepsilon(4^{j+1})\pi(4^{j+1})^2$. Then

$$\text{area } Q \setminus D \geq n^2 - \varepsilon(4^{j+1})\pi(4^{j+1})^2 \geq n^2/2$$

for every $Q \subset B_{4^{j+1}}$. By Proposition 3.12,

$$\int_Q |\text{grad } f|^2 dx \geq \frac{\pi^2}{2} \cdot \frac{\text{area } Q \setminus D}{2\pi n^2} \cdot \frac{1}{2n^2} \int_Q f^2 dx \\ = \frac{\pi}{16n^2} \int_Q f^2 dx$$

for every continuously differentiable function f in Q which vanishes on $Q \setminus D$.

For $t \in [4^j, 2 \cdot 4^j]$, set $A = \{x; t < |x| < 2t\}$. Then $\bigcup \{Q_{nk_1 k_2}; Q_{nk_1 k_2} \subset A\}$ contains an annulus $A' = \{x; a(t)t < |x| < b(t)t\}$, where $1 < a(t) < b(t) < 2$. It follows that

$$\int_A |\text{grad } f|^2 dx \geq \frac{\pi}{16n^2} \int_{A'} f^2 dx.$$

By the remark to Lemma 3.13 and Corollary 3.15, to show $h^{(p)}(0, D) < \infty$, it is sufficient to show that

$$\frac{\pi}{16n^2} \geq \frac{2}{3C(t)} 16^{p-j},$$

where $C(t) = b(t)^2 - a(t)^2$. We note that $C(t) \rightarrow 2^2 - 1 = 3$ as $t \rightarrow \infty$. Set $M = M(j) = 4^j/n$. Then the condition on n implies that $M^2 = O(1/\varepsilon(4^{j+1}))$ and so, for every $p > 0$, we can find $j_0 = j_0(p)$ such that the required inequality holds for every $j > j_0$. Hence $h^{(p)}(0, D) < \infty$ for every $p > 0$.

From the proof, we see that it is not necessary to assume that $\text{area } B_t \cap D / (\pi t^2) \rightarrow 0$ as $t \rightarrow \infty$. It is enough to assume that there is a positive constant $a \leq 1$ and a sequence $\{n_j\}_{j=1}^{\infty}$ of natural numbers such that $M_j = 4^j/n_j \rightarrow \infty$ as $j \rightarrow \infty$ and $\text{area } Q \setminus D \geq an_j^2$ for every $Q = Q_{n_j k_1 k_2} \subset \{x; 4^j < |x| < 4^{j+1}\}$. In this case, $\limsup_{t \rightarrow \infty} \text{area } B_t \cap D / (\pi t^2) \leq 1 - a$ and it is easy to construct an example for which equality holds.

3. *Exit times of Brownian motion in \mathbf{R}^d .* Let D be a domain in \mathbf{R}^d , let X be a Brownian motion in \mathbf{R}^d starting at a point x in D , and let τ be the first exit time of X from D , namely, $\tau(\omega) = \inf\{t > 0; X_t(\omega) \notin D\}$, where $\tau(\omega) = \infty$ if the set of the right-hand side is empty.

If D is bounded, the mean exit time $m(x) = E_x \tau$ of X tends to 0 as x approaches to a regular boundary point of D and it is a positive solution of the Poisson equation $\Delta m = -2$ in D , see Dynkin and Yushkevich [D-Y, pp. 68–69]. Hence $dm(x) + |x|^2$ is harmonic in D and it is the least harmonic majorant of $|x|^2$, namely, $h^{(2)}(x, D) = dm(x) + |x|^2$. By taking an exhaustion, we see that the equality also holds for an unbounded domain, nevertheless the boundary values of $m(x)$ are not equal to zero, see Burkholder [Bu 2, §4]. It can be expressed as $E_x |X_\tau|^2 = E_x(d\tau + |x|^2)$ and so there is a possibility of finding a relation between $E_x |X_\tau|^p = h^{(p)}(x, D)$ and $E_x(d\tau + |x|^2)^{p/2}$.

Burkholder [Bu 1] did obtain a relation. He showed

$$C_1 E_x(d\tau + |x|^2)^{p/2} \leq h^{(p)}(x, D) \leq C_2 E_x(d\tau + |x|^2)^{p/2},$$

where C_1 and C_2 are constants depending only on d and p . In particular, a moment $E_x \tau^{p/2}$ is finite if and only if $h^{(p)}(x, D)$ is finite. If we combine our results with that of Burkholder, we have many estimations of $E_x \tau^{p/2}$. Here we give an example.

THEOREM 8.3. *For an arbitrary domain D in \mathbf{R}^d , it follows that*

$$E_x \tau \leq \frac{2}{d\sigma} \int_D \frac{1}{|y - x|^{d-2}} dy.$$

PROOF. Take a new coordinate so that x becomes the origin. Then $E_0 \tau = m(0) = (1/d)h^{(2)}(0, D - x)$, where $D - x = \{y - x; y \in D\}$. Hence the proposition follows from Lemma 1.3. Q.E.D.

In Theorem 8.3, we have applied just Lemma 1.3. If we apply Proposition 6.3 or other estimations of $\omega_t(0)$, we have more accurate estimations of $E_x \tau$.

4. *Solutions of the Poisson equation.* We consider the inhomogeneous problem

$$\begin{aligned} \Delta u + f &= 0 & \text{if } D, \\ u &= 0 & \text{on } \partial D, \end{aligned}$$

where D is a bounded domain in \mathbf{R}^d . By using the volume radius $r(D)$ of D , Payne [Pa] gave an estimation of $u(x)$ when f is bounded and Weinberger [W] gave another one when f is of $L^p(D)$ for $p > d/2$. We apply Theorem 6.4 to the problem and show the following:

THEOREM 8.4. *It follows that*

(1) *If $x_0 \in D$ and if $f(x) \leq A/|x - x_0|^q$ in D with $0 \leq q < 2$ ($0 \leq q \leq 1$ if $d = 1$) and $A \geq 0$, then*

$$u(x_0) \leq \frac{A}{(2-q)(d-q)} r(D)^{2-q}.$$

(2) *If $x_0 \in D$ and if $f(x) \leq A|x - x_0|^p$ in D with $0 \leq p < \infty$ and $A \geq 0$, then*

$$u(x_0) \leq \frac{Ac(p+2)^{p+2}}{(p+2)(p+d)} r(D)^{p+2},$$

where $c(p+2)$ is a constant depending only on d and p and $c(p+2) = 1$ if $0 \leq p \leq (d-2) + 2^{1-d}$.

PROOF. If U is a solution of

$$\begin{aligned}\Delta U + F &= 0 && \text{in } D, \\ U &= 0 && \text{on } \partial D,\end{aligned}$$

where $F \geq f$ in D , then $\Delta(U - u) = f - F \leq 0$ in D and $U - u = 0$ on ∂D . Hence $U - u$ is superharmonic and nonnegative in D . In particular, $u(x_0) \leq U(x_0)$. Let

$$U(x) = \frac{A(h(x) - |x - x_0|^{p+2})}{(p+2)(p+d)},$$

where h is the least harmonic majorant of $|x - x_0|^{p+2}$ in D . Then

$$\begin{aligned}\Delta U + A|x - x_0|^p &= 0 && \text{in } D, \\ U &= 0 && \text{on } \partial D,\end{aligned}$$

and so $u(x_0) \leq U(x_0) = A((p+2)(p+d))^{-1}h(x_0)$. Therefore, the theorem follows from Theorem 6.4. Q.E.D.

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NOTE ADDED IN PROOF. M. Essén has pointed out that C. S. Stanton gave a new proof of the Alexander-Taylor-Ullman inequality in his recent paper: *Counting functions and majorization for Jensen measures* (preprint) and that there are applications of the inequality in a paper written by S. Axler and J. H. Shapiro: *Putnam's theorem, Alexander's spectral area estimate, and VMO*, Math. Ann. **271** (1985), 161–183. K. Haliste has pointed out that S. Friedland and W. K. Hayman proved an estimate similar to or, in a sense, better than that given in Lemma 3.4: *Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions*, Comment. Math. Helv. **51** (1976), 133–161. They made use of Huber's results, whereas our estimation was proved directly and elementarily.

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